George E. Andrews
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Ramanujan's Lost Notebook

Part II

Springer

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S. Ramanujan

George E. Andrews • Bruce C. Berndt

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By the kindness of heaven,
O lovely faced one,
You stand before me,
The darkness of delusion dispelled,
By recollection of that which was lost.

Verse 7.22 of Kalidasa's *Sakuntala*,
4th century A.D.

Preface

This is the second of approximately four volumes that the authors plan to write in their examination of all the claims made by S. Ramanujan in *The Lost Notebook and Other Unpublished Papers*. This volume, published by Narosa in 1988, contains the "Lost Notebook," which was discovered by the first author in the spring of 1976 at the library of Trinity College, Cambridge. Also included in this publication are other partial manuscripts, fragments, and letters that Ramanujan wrote to G.H. Hardy from nursing homes during 1917–1919. The authors have attempted to organize this disparate material in chapters. This second volume contains 16 chapters comprising 314 entries, including some duplications and examples, with chapter totals ranging from a high of fifty-four entries in Chapter 1 to a low of two entries in Chapter 12.

Contents

Introduction

This volume is the second of approximately four volumes that the authors plan to write on Ramanujan's lost notebook. We broadly interpret "lost notebook" to include all material published *with* Ramanujan's original lost notebook by Narosa in 1988 [244]. Thus, when we write that a certain entry is found in the lost notebook, it may not actually be located in the original lost notebook discovered by the first author in the spring of 1976 at Trinity College Library, Cambridge, but instead may be in a manuscript, fragment, or a letter of Ramanujan to G.H. Hardy published in [244]. We are attempting to arrange all this disparate material into chapters for each of the proposed volumes. For a history and general description of Ramanujan's lost notebook, readers are advised to read the introduction to our first book [31].

The Organization of Entries

With the statement of each entry from Ramanujan's lost notebook, we provide the page number(s) in the lost notebook on which the entry can be found. All of Ramanujan's claims are given the designation "Entry." Results in this volume named theorems, corollaries, and lemmas are (unless otherwise stated) not due to Ramanujan. We emphasize that Ramanujan's claims always have page numbers from the lost notebook attached to them. We remark that in Chapter 9, which is devoted to establishing Ramanujan's values for an analogue λ_n of the classical Ramanujan–Weber class invariant G_n, we have followed a slightly different convention. Indeed, we have listed all of Ramanujan's values for λ_n in Entry 9.1.1 with the page number indicated. Later, we establish these values as corollaries of theorems that we prove, and so we record Ramanujan's values of λ_n *again*, listing them as corollaries with page numbers in the lost notebook attached to emphasize that these corollaries are due to Ramanujan.

In view of the subject mentioned in the preceding paragraph, it may be prudent to make a remark here about Ramanujan's methods. As many read-

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ers are aware from the work of the authors and others who have attempted to prove Ramanujan's theorems, we frequently have few or no clues about Ramanujan's methods. Many of the proofs of the values for G_n that are given in [57] are almost certainly not those found by Ramanujan, for he would have needed knowledge of certain portions of mathematics that he likely did not know or that had not been discovered yet. Similar remarks can be made about our calculations of λ_n in Chapter 9. In the last half of the chapter, we employ ideas that Ramanujan would not have known.

So that readers can more readily find where a certain entry from the lost notebook is discussed, we place at the conclusion of each volume a *Location Guide* indicating where entries can be found in that particular volume. Thus, for example, if a reader wants to know whether a certain identity on page 1729 of the Narosa edition [244] can be found in a particular volume, she can turn to this index and determine where in that volume identities on page 1729 are discussed.

Following the Location Guide, we provide a *Provenance* indicating the sources from which we have drawn in preparing significant portions of the given chapters. We emphasize that in the Provenance we do not list all papers in which results from a given chapter are established. For example, in Chapter 3, Ramanujan's famous ${}_1\psi_1$ summation theorem, which is found in more than one version in the lost notebook, is discussed, but we do not refer to all papers on the ${}_1\psi_1$ summation formula in the Location Guide, although in Chapter 3 itself, we have attempted to cite all relevant proofs of this celebrated formula. On the other hand, most chapters contain previously unpublished material. For example, each of the first four chapters contains previously unpublished proofs.

This Volume on the Lost Notebook

Two primary themes permeate our second volume on the lost notebook, namely, q-series and Eisenstein series. The first seven chapters are devoted to q-series identities from the core of the original lost notebook. These chapters are followed by three chapters on identities for the classical theta functions or related functions. The last six chapters feature Eisenstein series, with much of the material originating in letters to Hardy that Ramanujan wrote from Fitzroy House and Matlock House during his last two years in England. We now briefly describe the contents of the sixteen chapters in this volume.

Heine's transformations have long been central to the theory of basic hypergeometric series. In Chapter 1, we examine several entries from the lost notebook that have their roots in Heine's first transformation or generalizations thereof. The Sears–Thomae transformation is also a staple in the theory of basic hypergeometric series, and consequences of it form the content of Chapter 2. In Chapter 3, we consider identities arising from certain bilateral series identities, in particular the renowned ${}_1\psi_1$ summation of Ramanujan and well-known identities due to W.N. Bailey. We have also placed in Chapter 3 some identities dependent upon the quintuple product identity. Watson's q-analogue of Whipple's theorem and two additional theorems of Bailey are the main ingredients for the proofs in Chapter 4 on well-poised series. Bailey's lemma is utilized to prove some identities in Chapter 5. Chapter 6, on partial theta functions, is one of the more difficult chapters in this volume. Chapter 7 contains entries from the lost notebook that are even more difficult to prove than those in Chapter 6. The entries in this chapter do not fall into any particular categories and bear further study, because several of them likely have yet-to-be discovered ramifications.

Theta functions frequently appear in identities in the first seven chapters. However, in Chapters 8–10, theta functions are the focus. Chapter 8 is devoted to theta function identities. Chapter 9 focuses on one page in the lost notebook on values of an analogue of the classical Ramanujan–Weber class invariants. The identities in Chapter 10 do not fit in any of the previous chapters and are among the most unusual identities we have seen in Ramanujan's work.

As remarked above, the last six chapters in this volume feature Eisenstein series. Perhaps the most important chapter is Chapter 11, which contains proofs of results sent to Hardy from nursing homes, probably in 1918. In these letters, Ramanujan offered formulas for the coefficients of certain quotients of Eisenstein series that are analogous to the Hardy–Ramanujan–Rademacher series representation for the partition function $p(n)$. The claims in these letters continue the work found in Hardy and Ramanujan's last joint paper [177], [242, pp. 310–321]. Chapter 12 relates technical material on the number of terms that one needs to take from the aforementioned series in order to determine these coefficients precisely. In Chapter 13, the focus shifts to identities for Eisenstein series involving the Dedekind eta function. Chapter 14 gives formulas for certain series associated with the pentagonal number theorem in terms of Ramanujan's Eisenstein series P, Q, and R. These results are found on two pages of the lost notebook, and, although not deep, have recently generated several further papers. Chapter 15 is devoted primarily to a single page in the lost notebook demonstrating how Ramanujan employed Eisenstein series to approximate π. Three series for $1/\pi$ found in Ramanujan's epic paper [239], [242, pp. 23–39] are also found on page 370 of [244], and so it seems appropriate to prove them in this chapter, especially since, perhaps more so than other authors, we follow Ramanujan's hint in [239] and use Eisenstein series to establish these series representations for $1/\pi$. This volume concludes with a few miscellaneous results on Eisenstein series in Chapter 16.

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1

The Heine Transformation

1.1 Introduction

E. Heine [178], [179, pp. 97–125] was the first to generalize Gauss's hypergeometric series to q-hypergeometric series by defining, for $|q| < 1$,

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, t\right) := \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(q;q)_n (c;q)_n} t^n, \tag{1.1.1} $$

where $|t| < 1$ and where, for each nonnegative integer n,

$$ (a)_n = (a;q)_n := (1-a)(1-aq)\cdots(1-aq^{n-1}), \tag{1.1.2} $$

with the convention that $(a)_0 = (a;q)_0 := 1$. If an entry and its proof involve only the base q and no confusion would arise, we use the notation at the left in (1.1.2) and (1.1.4) below. If more than one base occurs in an entry and/or its proof, e.g., both q and q^2 appear, then we use the second notation in (1.1.2) and (1.1.4). Ramanujan's central theorem is a transformation for this series, now known as the Heine transformation, namely [179, p. 106, equation (50)],

$$ {}_2\phi_1\left(\begin{matrix} a, b \\ c \end{matrix}; q, t\right) = \frac{(b;q)_\infty (at;q)_\infty}{(c;q)_\infty (t;q)_\infty} {}_2\phi_1\left(\begin{matrix} c/b, t \\ at \end{matrix}; q, b\right), \tag{1.1.3} $$

where $|t|, |b| < 1$ and where

$$ (a)_\infty = (a;q)_\infty = \lim_{n\to\infty} (a;q)_n, \qquad |q| < 1. \tag{1.1.4} $$

His method of proof was surely known to Ramanujan, who recorded an equivalent formulation of (1.1.3) in Entry 6 of Chapter 16 in his second notebook [243], [54, p. 15]. Furthermore, numerous related identities can be proved using Heine's original idea.

In Section 1.2, we prove several basic formulas based on Heine's method. In the remainder of the chapter we deduce 53 formulas found in the lost

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notebook. In some instances, we call upon a result not listed in Section 1.2, but each identity that we prove relies primarily on results in Section 1.2.

In order to keep our proofs to manageable lengths, we invoke certain standard simplifications (usually without mentioning them explicitly), such as

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}, \tag{1.1.5}$$

$$(a;q)_n(-a;q)_n = (a^2;q^2)_n, \qquad 0 \le n < \infty, \tag{1.1.6}$$

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty}, \qquad -\infty < n < \infty. \tag{1.1.7}$$

The identity (1.1.5) is a famous theorem of Euler, which we invoke numerous times in this book. Identity (1.1.7) can be regarded as the definition of $(a;q)_n$ when n is a negative integer.

1.2 Heine's Method

In [6], Heine's method was encapsulated in a fundamental formula containing ten independent variables and a nontrivial root of unity. As a result, it is an almost unreadable formula. Consequently, we prove only special cases of this result here. In light of the fact that many of these results are not easily written in the notation (1.1.1) of q-hypergeometric series, we record all our results in terms of infinite series. For further work connected with that of Andrews in [6], see Z. Cao's thesis [97] and a paper by W. Chu and W. Zhang [131].

We begin with a slightly generalized version of Heine's transformation [6], [7].

Theorem 1.2.1. *If h is a positive integer, then, for $|t|, |b| < 1$,*

$$\sum_{m=0}^{\infty} \frac{(a;q^h)_m(b;q)_{hm}}{(q^h;q^h)_m(c;q)_{hm}} t^m = \frac{(b;q)_\infty(at;q^h)_\infty}{(c;q)_\infty(t;q^h)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m(t;q^h)_m}{(q;q)_m(at;q^h)_m} b^m. \tag{1.2.1}$$

Proof. We need the q-binomial theorem given by [54, p. 14, Entry 2], [18, p. 17, Theorem 2.1]

$$\sum_{m=0}^{\infty} \frac{(a/b;q)_m}{(q;q)_m} b^m = \frac{(a;q)_\infty}{(b;q)_\infty}, \tag{1.2.2}$$

where $|b| < 1$. Since we frequently need two special cases in the sequel, we state them here. If $a = 0$ in (1.2.2), then [18, p. 19, equation (2.2.5)]

$$\sum_{m=0}^{\infty} \frac{b^m}{(q;q)_m} = \frac{1}{(b;q)_\infty}. \tag{1.2.3}$$

Letting $b \to 0$ in (1.2.2), we find that [18, p. 19, equation (2.2.6)]

$$\sum_{m=0}^{\infty} \frac{(-a)^m q^{m(m-1)/2}}{(q;q)_m} = (a;q)_\infty. \tag{1.2.4}$$

Upon two applications of (1.2.2), we see that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(a;q^h)_n (b;q)_{hn}}{(q^h;q^h)_n (c;q)_{hn}} t^n &= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^h)_n}{(q^h;q^h)_n} \frac{(cq^{hn};q)_\infty}{(bq^{hn};q)_\infty} t^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^h)_n}{(q^h;q^h)_n} t^n \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m} b^m q^{hmn} \\
&= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m} b^m \sum_{n=0}^{\infty} \frac{(a;q^h)_n}{(q^h;q^h)_n} (tq^{hm})^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m} b^m \frac{(atq^{hm};q^h)_\infty}{(tq^{hm};q^h)_\infty} \\
&= \frac{(b;q)_\infty (at;q^h)_\infty}{(c;q)_\infty (t;q^h)_\infty} \sum_{m=0}^{\infty} \frac{(c/b;q)_m (t;q^h)_m}{(q;q)_m (at;q^h)_m} b^m,
\end{aligned}$$

which is (1.2.1). □

Heine's transformation is the case $h = 1$ of Theorem 1.2.1, and Theorem A_3 of [6] is the case $h = 2$. The complete result appears in [7, Lemma 1].

The next result is more intricate, but it is based again on Heine's idea; it is Theorem A_1 of [6].

Theorem 1.2.2. *For* $|t|, |b| < 1$,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(a;q^2)_n (b;q)_n}{(q^2;q^2)_n (c;q)_n} t^n = \frac{(b;q)_\infty (at;q^2)_\infty}{(c;q)_\infty (t;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(c/b;q)_{2n} (t;q^2)_n}{(q;q)_{2n} (at;q^2)_n} b^{2n} \\
+ \frac{(b;q)_\infty (atq;q^2)_\infty}{(c;q)_\infty (tq;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(c/b;q)_{2n+1} (tq;q^2)_n}{(q;q)_{2n+1} (atq;q^2)_n} b^{2n+1}.
\end{aligned} \tag{1.2.5}$$

Proof. Using (1.2.2) twice, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(a;q^2)_n (b;q)_n}{(q^2;q^2)_n (c;q)_n} t^n = \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^2)_n}{(q^2;q^2)_n} \frac{(cq^n;q)_\infty}{(bq^n;q)_\infty} t^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^2)_n}{(q^2;q^2)_n} t^n \sum_{m=0}^{\infty} \frac{(c/b;q)_m}{(q;q)_m} b^m q^{mn} \\
&= \frac{(b;q)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(a;q^2)_n}{(q^2;q^2)_n} t^n \left\{ \sum_{m=0}^{\infty} \frac{(c/b;q)_{2m}}{(q;q)_{2m}} b^{2m} q^{2mn} \right. \\
&\left. + \sum_{m=0}^{\infty} \frac{(c/b;q)_{2m+1}}{(q;q)_{2m+1}} b^{2m+1} q^{(2m+1)n} \right\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^{\infty}\frac{(c/b;q)_{2m}}{(q;q)_{2m}}b^{2m}\sum_{n=0}^{\infty}\frac{(a;q^2)_n}{(q^2;q^2)_n}(tq^{2m})^n \\
&\quad + \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^{\infty}\frac{(c/b;q)_{2m+1}}{(q;q)_{2m+1}}b^{2m+1}\sum_{n=0}^{\infty}\frac{(a;q^2)_n}{(q^2;q^2)_n}(tq^{2m+1})^n \\
&= \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^{\infty}\frac{(c/b;q)_{2m}}{(q;q)_{2m}}b^{2m}\frac{(atq^{2m};q^2)_\infty}{(tq^{2m};q^2)_\infty} \\
&\quad + \frac{(b;q)_\infty}{(c;q)_\infty}\sum_{m=0}^{\infty}\frac{(c/b;q)_{2m+1}}{(q;q)_{2m+1}}b^{2m+1}\frac{(atq^{2m+1};q^2)_\infty}{(tq^{2m+1};q^2)_\infty} \\
&= \frac{(b;q)_\infty(at;q^2)_\infty}{(c;q)_\infty(t;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(c/b;q)_{2m}(t;q^2)_m}{(q;q)_{2m}(at;q^2)_m}b^{2m} \\
&\quad + \frac{(b;q)_\infty(atq;q^2)_\infty}{(c;q)_\infty(tq;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(c/b;q)_{2m+1}(tq;q^2)_m}{(q;q)_{2m+1}(atq;q^2)_m}b^{2m+1}.
\end{aligned}
$$

□

In addition to Theorems 1.2.1 and 1.2.2, we require two corollaries of Theorem 1.2.1. The first is also given in [7, equation (I5)].

Corollary 1.2.1. *For* $|t| < 1$,

$$
\sum_{n=0}^{\infty}\frac{(b;q)_{2n}}{(q^2;q^2)_n}t^{2n} = \frac{(-tb;q)_\infty}{(-t;q)_\infty}\sum_{n=0}^{\infty}\frac{(b;q)_n}{(q;q)_n(-tb;q)_n}t^n. \tag{1.2.6}
$$

Proof. By (1.2.1) with $h = 2$, $a = c = 0$, and t replaced by t^2, we see that

$$
\begin{aligned}
\sum_{n=0}^{\infty}\frac{(b;q)_{2n}}{(q^2;q^2)_n}t^{2n} &= \frac{(b;q)_\infty}{(t^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(t^2;q^2)_n}{(q;q)_n}b^n \\
&= \frac{(b;q)_\infty}{(t^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(t;q)_n(-t;q)_n}{(q;q)_n}b^n \\
&= \frac{(b;q)_\infty}{(t^2;q^2)_\infty}\frac{(t;q)_\infty(-tb;q)_\infty}{(b;q)_\infty}\sum_{n=0}^{\infty}\frac{(b;q)_n}{(q;q)_n(-tb;q)_n}t^n,
\end{aligned}
$$

by (1.2.1) with $t = b$ and then $h = 1$, $a = -t$, $b = t$, and $c = 0$. Upon simplification above, we deduce (1.2.6). □

The next result can be found in [7, equation (I6)].

Corollary 1.2.2. *For* $|b| < 1$,

$$
\sum_{n=0}^{\infty}\frac{(t;q^2)_n}{(q;q)_n}b^n = \frac{(btq;q^2)_\infty}{(bq;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(t;q^2)_n}{(q^2;q^2)_n(btq;q^2)_n}b^n. \tag{1.2.7}
$$

Proof. By (1.2.1) with $h = 2$ and $a = c = 0$, we see that

$$\begin{aligned}\frac{(b;q)_\infty}{(t;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(t;q^2)_n}{(q;q)_n}b^n &= \sum_{n=0}^{\infty}\frac{(b;q)_{2n}}{(q^2;q^2)_n}t^n\\ &= \sum_{n=0}^{\infty}\frac{(bq;q^2)_n(b;q^2)_n}{(q^2;q^2)_n}t^n\\ &= \frac{(b;q^2)_\infty(btq;q^2)_\infty}{(t;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(t;q^2)_n}{(q^2;q^2)_n(btq;q^2)_n}b^n,\end{aligned}$$

where we applied (1.2.1) with q replaced by q^2, $h = 1$, $a = bq$, and $c = 0$. Upon simplification, we complete the proof. □

Our next result comes from [9, Theorem 7].

Corollary 1.2.3. *For* $|t| < 1$,

$$\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q^2)_n}{(q;q)_n(abt;q^2)_n}t^n = \frac{(at;q^2)_\infty(bt;q^2)_\infty}{(t;q^2)_\infty(abt;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(a;q^2)_n(b;q^2)_n}{(q^2;q^2)_n(bt;q^2)_n}(tq)^n. \tag{1.2.8}$$

Proof. In (1.2.1), set $h = 2$, interchange t with b, replace a by at, and then replace c by at. Upon simplification, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q^2)_n}{(q;q)_n(abt;q^2)_n}t^n &= \frac{(at;q)_\infty(b;q^2)_\infty}{(t;q)_\infty(abt;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(at;q^2)_n(t;q)_{2n}}{(q^2;q^2)_n(at;q)_{2n}}b^n\\ &= \frac{(at;q)_\infty(b;q^2)_\infty}{(t;q)_\infty(abt;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(t;q^2)_n(tq;q^2)_n}{(q^2;q^2)_n(atq;q^2)_n}b^n\\ &= \frac{(at;q)_\infty(b;q^2)_\infty(tq;q^2)_\infty(bt;q^2)_\infty}{(t;q)_\infty(abt;q^2)_\infty(atq;q^2)_\infty(b;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(a;q^2)_n(b;q^2)_n}{(q^2;q^2)_n(bt;q^2)_n}(tq)^n,\end{aligned}$$

where we invoked (1.2.1) with $h = 1$, q replaced by q^2, and the variables a, b, c, and t replaced by t, tq, atq, and b, respectively. Upon simplifying above, we deduce (1.2.8) to complete the proof. □

We also require the direct iteration of (1.2.1) with $h = 1$ [9, Theorem 8]. This is often called the second Heine transformation.

Corollary 1.2.4. *For* $|t|, |c/b| < 1$,

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(q)_n(c)_n}t^n = \frac{(c/b)_\infty(bt)_\infty}{(c)_\infty(t)_\infty}\sum_{n=0}^{\infty}\frac{(abt/c)_n(b)_n}{(q)_n(bt)_n}\left(\frac{c}{b}\right)^n. \tag{1.2.9}$$

Proof. By two applications of Theorem 1.2.1 with $h = 1$, the second with a, b, c, and t replaced by t, c/b, at, and b, respectively, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(q)_n(c)_n} t^n &= \frac{(b)_\infty(at)_\infty}{(c)_\infty(t)_\infty} \sum_{n=0}^{\infty} \frac{(c/b)_n(t)_n}{(q)_n(at)_n} b^n \\ &= \frac{(b)_\infty(at)_\infty}{(c)_\infty(t)_\infty} \frac{(c/b)_\infty(bt)_\infty}{(at)_\infty(b)_\infty} \sum_{n=0}^{\infty} \frac{(abt/c)_n(b)_n}{(q)_n(bt)_n} \left(\frac{c}{b}\right)^n,\end{aligned}$$

which is the desired result. □

Finally, we need one more iteration of (1.2.1) with $h = 1$ [18, p. 39, equation (3.3.13)]. This is often called the q-analogue of Euler's transformation.

Corollary 1.2.5. *For* $|t|, |abt/c| < 1$,

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(q)_n(c)_n} t^n = \frac{(abt/c)_\infty}{(t)_\infty} \sum_{n=0}^{\infty} \frac{(c/a)_n(c/b)_n}{(q)_n(c)_n} \left(\frac{abt}{c}\right)^n. \tag{1.2.10}$$

Proof. Apply (1.2.1) with $h = 1$ and a, b, c, and t replaced by b, abt/c, bt, and c/b, respectively. Consequently,

$$\sum_{n=0}^{\infty} \frac{(abt/c)_n(b)_n}{(q)_n(bt)_n} \left(\frac{c}{b}\right)^n = \frac{(abt/c)_\infty(c)_\infty}{(bt)_\infty(c/b)_\infty} \sum_{n=0}^{\infty} \frac{(c/a)_n(c/b)_n}{(q)_n(c)_n} \left(\frac{abt}{c}\right)^n. \tag{1.2.11}$$

Substituting the right-hand side of (1.2.11) for the sum on the right-hand side of (1.2.9) and simplifying yields (1.2.10). □

1.3 Ramanujan's Proof of the q-Gauss Summation Theorem

On pages 268–269 in his lost notebook, Ramanujan sketches his proof of the q-Gauss summation theorem, normally given in the form

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(q)_n} \left(\frac{c}{ab}\right)^n = \frac{(c/a)_\infty(c/b)_\infty}{(c)_\infty(c/(ab))_\infty}. \tag{1.3.1}$$

This theorem was first discovered in 1847 by Heine [178], whose proof, which is the most frequently encountered proof in the literature, is based on Heine's transformation, Theorem 1.2.1, with $h = 1$. This proof can be found in the texts of Andrews [18, p. 20, Corollary 2.4], Andrews, R. Askey, and R. Roy [30, p. 522], and G. Gasper and M. Rahman [151, p. 10]. A second proof employs the q-analogue of Saalschütz's theorem and can be read in the texts of W.N. Bailey [44, p. 68] and L.J. Slater [263, p. 97]. Ramanujan's proof is different from these two proofs and was first published in full in a paper by Berndt and A.J. Yee [79]. Ramanujan's proof encompasses Lemma 1.3.1, Lemma 1.3.2, and Entry 1.3.1 below. After giving Ramanujan's proof, we prove a corollary of (1.3.1), which is found on page 370 in Ramanujan's lost notebook.

Before providing Ramanujan's argument, we derive the q-analogue of the Chu–Vandermonde theorem and record a special case that will be used in Chapter 6. If we set $b = q^{-N}$, where N is a nonnegative integer, in (1.3.1) and simplify, we find that

$$ {}_2\phi_1(a, q^{-N}; c; q, cq^N/a) = \frac{(c/a)_N}{(c)_N}, \tag{1.3.2} $$

which is the q-analogue of the Chu–Vandermonde theorem. If we reverse the order of summation on the left-hand side of (1.3.2), we deduce an alternative form of the q-Chu–Vandermonde theorem, namely,

$$ {}_2\phi_1(a, q^{-N}; c; q, q) = \frac{(c/a)_N}{(c)_N} a^N. \tag{1.3.3} $$

Setting $a = q^{-M}$ and $c = q^{-M-N}$, where M is a nonnegative integer, in (1.3.3) yields

$$ \begin{aligned} {}_2\phi_1(q^{-M}, q^{-N}; q^{-M-N}; q, q) &= \frac{(q^{-N})_N}{(q^{-M-N})_N} q^{-MN} = \frac{(q^{-M})_M (q^{-N})_N}{(q^{-M-N})_{M+N}} q^{-MN} \\ &= \frac{(q)_M (q)_N q^{-M(M+1)/2 - N(N+1)/2}}{(q)_{M+N} q^{-(M+N)(M+N+1)/2}} q^{-MN} \\ &= \frac{(q)_M (q)_N}{(q)_{M+N}}. \end{aligned} \tag{1.3.4} $$

In this chapter, we are providing analytic proofs of many of Ramanujan's theorems on basic hypergeometric series. Another approach uses combinatorial arguments. In [78], Berndt and Yee provided partition-theoretic proofs of several identities in the lost notebook arising from the Rogers–Fine identity; a few of these proofs were reproduced in [31, Chapter 12]. In [79], the same authors gave a combinatorial proof of the q-Gauss summation theorem. Other combinatorial proofs of this theorem based on overpartitions have been given by S. Corteel and J. Lovejoy [144], Corteel [143], and Yee [285].

Lemma 1.3.1. *If n is any nonnegative integer, then*

$$ (a)_n = \sum_{k=0}^{n} (-1)^k \frac{(q^{n+1-k})_k}{(q)_k} q^{k(k-1)/2} a^k. \tag{1.3.5} $$

Lemma 1.3.1 is a restatement of the q-binomial theorem (1.2.2) and can be found in [54, p. 24, Lemma 12.1] or [18, p. 36, Theorem 3.3]. We now use Lemma 1.3.1 to establish Lemma 1.3.2 below along the lines indicated by Ramanujan. Alternatively, Lemma 1.3.2 can be deduced from [151, p. 11, equation (1.5.3)] by setting $c = 0$ and replacing q by $1/q$ there.

Lemma 1.3.2. *If $c \neq 0$ and n is any nonnegative integer, then*

$$c^n = \sum_{j=0}^{n} \frac{c^j (1/c)_j (q^{n+1-j})_j}{(q)_j}. \tag{1.3.6}$$

Proof. Denote the right side of (1.3.6) by $g(c)$ and apply (1.3.5) with $a = 1/c$ and $n = j$ in the definition of $g(c)$ to find that

$$g(c) = \sum_{j=0}^{n} \sum_{k=0}^{j} (-1)^k \frac{(q^{j+1-k})_k (q^{n+1-j})_j}{(q)_j (q)_k} q^{k(k-1)/2} c^{j-k} =: \sum_{r=0}^{n} a_r c^r.$$

The coefficient of $c^r, 0 \leq r \leq n$, above is

$$a_r = \sum_{k=0}^{n-r} (-1)^k \frac{(q^{r+1})_k (q^{n+1-r-k})_{r+k}}{(q)_{r+k} (q)_k} q^{k(k-1)/2}. \tag{1.3.7}$$

Now we can easily verify that

$$\frac{(q^{r+1})_k}{(q)_{r+k}} = \frac{1}{(q)_r}$$

and

$$(q^{n+1-r-k})_{r+k} = (q^{n+1-r-k})_k (q^{n+1-r})_r.$$

Using these last two equalities in (1.3.7), we find that

$$\begin{aligned} a_r &= \frac{(q^{n+1-r})_r}{(q)_r} \sum_{k=0}^{n-r} (-1)^k \frac{(q^{n-r+1-k})_k}{(q)_k} q^{k(k-1)/2} \\ &= \frac{(q^{n+1-r})_r}{(q)_r} (1)_{n-r} = \begin{cases} 1, & \text{if } r = n, \\ 0, & \text{otherwise}, \end{cases} \end{aligned}$$

by (1.3.5). This therefore completes our proof of Lemma 1.3.2. □

Entry 1.3.1 (pp. 268–269, q-Gauss Summation Theorem). *If $|abc| < 1$ and $bc \neq 0$, then*

$$\frac{(ac)_\infty}{(abc)_\infty} = \frac{(a)_\infty}{(ab)_\infty} \sum_{n=0}^{\infty} \frac{(1/b)_n (1/c)_n}{(a)_n (q)_n} (abc)^n. \tag{1.3.8}$$

In Entry 4 of Chapter 16 in his second notebook [243], [54, p. 14], Ramanujan states the q-Gauss summation theorem in precisely the same form as that given in (1.3.8).

Proof. We rewrite the right side of (1.3.8) in the form

$$\sum_{j=0}^{\infty} \frac{(aq^j)_\infty}{(ab)_\infty} \frac{(1/b)_j(1/c)_j}{(q)_j}(abc)^j \tag{1.3.9}$$

and examine the coefficient of a^n, $n \geq 0$, on each side of (1.3.8). From (1.2.2), with b replaced by ab and a replaced by aq^j, we find that

$$\frac{(aq^j)_\infty}{(ab)_\infty} = \sum_{k=0}^{\infty} \frac{(q^j/b)_k}{(q)_k}(ab)^k. \tag{1.3.10}$$

The coefficient of a^{n-j} in (1.3.10) is

$$\frac{(q^j/b)_{n-j}}{(q)_{n-j}} b^{n-j},$$

and so the coefficient of a^n in (1.3.9) equals

$$\sum_{j=0}^{n} \frac{(1/b)_j(1/c)_j(q^j/b)_{n-j}}{(q)_j(q)_{n-j}} b^n c^j$$

$$= \frac{(1/b)_n b^n}{(q)_n} \sum_{j=0}^{n} \frac{c^j(1/c)_j(q^{n+1-j})_j}{(q)_j} = \frac{(1/b)_n b^n}{(q)_n} c^n, \tag{1.3.11}$$

by Lemma 1.3.2. But by (1.2.2), with b replaced by abc and a replaced by ac,

$$\frac{(ac)_\infty}{(abc)_\infty} = \sum_{n=0}^{\infty} \frac{(1/b)_n}{(q)_n}(abc)^n. \tag{1.3.12}$$

So, the coefficient of a^n in (1.3.12) is precisely that on the right side of (1.3.11). Hence, (1.3.8) immediately follows, since the coefficients of a^n, $n \geq 0$, on both sides of (1.3.8) are equal. The proof of Entry 1.3.1 is therefore complete. □

Entry 1.3.2 (p. 370). *For any complex numbers a and b,*

$$\frac{(-aq)_\infty}{(bq)_\infty} = \sum_{n=0}^{\infty} \frac{(-b/a)_n a^n q^{n(n+1)/2}}{(q)_n(bq)_n}. \tag{1.3.13}$$

Proof. In (1.3.8), replace a by bq, c by $-a/b$, and b by t to find that

$$\frac{(bqt)_\infty(-aq)_\infty}{(bq)_\infty(-aqt)_\infty} = \sum_{n=0}^{\infty} \frac{(1/t)_n(-b/a)_n}{(q)_n(bq)_n}(-aqt)^n. \tag{1.3.14}$$

If we let $t \to 0$ in (1.3.14), we immediately arrive at (1.3.13) to complete the proof. □

A combinatorial proof of Entry 1.3.2 in the case $b = 1$ has been given by S. Corteel and J. Lovejoy [145], but it can easily be extended to give a proof of Entry 1.3.2 in full generality. Another combinatorial proof can be found in a paper by Berndt, B. Kim, and A.J. Yee [73].

1.4 Corollaries of (1.2.1) and (1.2.5)

Entry 1.4.1 (p. 3). *For* $0 < |aq|, |k| < 1$,

$$\frac{(aq;q)_\infty (cq;q^2)_\infty}{(-bq;q)_\infty (kq^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(kq^2;q^2)_n(-bq/a;q)_n}{(cq;q^2)_{n+1}(q;q)_n} a^n q^n$$
$$= \sum_{n=0}^{\infty} \frac{(cq/k;q^2)_n (aq;q)_{2n}}{(q^2;q^2)_n(-bq;q)_{2n+1}} k^n q^{2n}. \qquad (1.4.1)$$

Proof. In (1.2.1), set $h=2$ and $t=kq^2$, and replace c by $-bq^2$, a by cq/k, and b by aq. The resulting identity is equivalent to (1.4.1). □

We note that no generality has been lost by the substitutions above; so Ramanujan had (1.2.1) in full generality for $h=2$. Padmavathamma [225] has also given a proof of (1.4.1).

Entry 1.4.2 (p. 3). *For* $|bq| < 1$,

$$(q;q^2)_\infty (aq;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-q;q)_n(-bq;q)_n}{(aq;q^2)_{n+1}} q^n$$
$$= (-bq;q)_\infty \sum_{n=0}^{\infty} \frac{(q;q^2)_n(aq;q^2)_n}{(-bq;q)_{2n+1}} q^{2n}.$$

Proof. In (1.2.1), set $h=2$, $b=q$, and $t=q^2$, and replace a by aq and c by $-bq^2$. The result then reduces to the identity above upon simplification. □

Entry 1.4.3 (p. 12). *For* $|aq|, |b| < 1$,

$$\sum_{n=0}^{\infty} \frac{a^n q^n}{(q;q)_n(bq;q^2)_n} = \frac{1}{(aq;q)_\infty(bq;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (aq;q)_{2n} b^n q^{n^2}}{(q^2;q^2)_n}.$$

Proof. In (1.2.1), set $h=2$, $c=0$, and $t=\tau$, and replace a by bq/τ and b by aq. Then let $\tau \to 0$. The result easily simplifies to the identity above. □

Entry 1.4.4 (p. 12). *For* $|a|, |b| < 1$,

$$\sum_{n=0}^{\infty} \frac{a^n q^{2n}}{(q^2;q^2)_n(bq;q)_{2n}} = \frac{1}{(aq^2;q^2)_\infty(bq;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (aq^2;q^2)_n b^n q^{n(n+1)/2}}{(q;q)_n}.$$

Proof. In (1.2.1), set $h=2$ and $a=0$, let $b \to 0$, and then replace t by aq^2 and c by bq. □

The previous two entries were also established by Padmavathamma [225]. The next result is a corrected version of Ramanujan's claim.

Entry 1.4.5 (p. 15, corrected). *For any complex number a,*

$$\sum_{n=0}^{\infty}(-aq;q)_n(-q;q)_n q^n = (-q;q)_\infty(-aq;q)_\infty \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n}}{(-aq;q)_{2n+1}}.$$

Proof. In (1.2.1), set $h=2$, $a=0$, $b=q$, $c=-aq^2$, and $t=q^2$. Simplification yields Ramanujan's assertion. □

The next two entries specialize to instances of identities for fifth-order mock theta functions, as we shall see in our fourth volume on the lost notebook [33]. The first is a corrected version of Ramanujan's claim.

Entry 1.4.6 (p. 16, corrected). *For any complex number a,*

$$\begin{aligned}\frac{(-aq;q)_\infty}{(-q;q)_\infty} &\sum_{n=0}^{\infty} \frac{(aq;q)_n q^{n^2}}{(q^2;q^2)_n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n(a^2q^2;q^2)_{2n} q^{2n^2}}{(q^4;q^4)_n} - a\sum_{n=1}^{\infty} \frac{(a^2q^2;q^2)_{n-1}(-q)^{n(n+1)/2}}{(-q;-q)_{n-1}}.\end{aligned}$$

Proof. The proof of this result is rather more intricate than the proofs of the previous entries in this section. In (1.2.5), replace t by $-q/a$ and let $a \to \infty$ to deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(b;q)_n q^{n^2}}{(q^2;q^2)_n(c;q)_n} &= \frac{(b;q)_\infty(-q;q^2)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(c/b)_{2n}}{(q;q)_{2n}(-q;q^2)_n} b^{2n} \\ &+ \frac{(b;q)_\infty(-q^2;q^2)_\infty}{(c;q)_\infty} \sum_{n=0}^{\infty} \frac{(c/b)_{2n+1}}{(q;q)_{2n+1}(-q^2;q^2)_n} b^{2n+1}.\end{aligned}$$

Now set $c=0$ and $b=aq$. If we multiply both sides of the resulting identity by $(-aq;q)_\infty/(-q;q)_\infty$, we arrive at

$$\begin{aligned}\frac{(-aq;q)_\infty}{(-q;q)_\infty} \sum_{n=0}^{\infty} \frac{(aq;q)_n q^{n^2}}{(q^2;q^2)_n} &= \frac{(a^2q^2;q^2)_\infty}{(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{a^{2n}q^{2n}}{(q;q)_{2n}(-q;q^2)_n} \\ &+ \frac{(a^2q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{a^{2n+1}q^{2n+1}}{(q;q)_{2n+1}(-q^2;q^2)_n} \\ &=: T_1 + T_2. \end{aligned} \tag{1.4.2}$$

Next, in (1.2.1) with $h=2$, replace q by q^2, set $a=q^2/t$, $b=a^2q^2$, and $c=0$, and let $t \to 0$. Noting that $(-q^2;q^2)_\infty = 1/(q^2;q^4)_\infty$, we deduce that

$$T_1 = \sum_{n=0}^{\infty} \frac{(-1)^n(a^2q^2;q^2)_{2n} q^{2n^2}}{(q^4;q^4)_n}. \tag{1.4.3}$$

Finally, in (1.2.1), set $h = 2$, $a = 0$, and $c = -q^2$, and let $b \to 0$. Then set $t = a^2q^2$ and multiply both sides of the resulting equality by $1/(1+q)$. We therefore find that

$$\sum_{n=0}^{\infty} \frac{a^{2n}q^{2n}}{(q^2;q^2)_n(-q;q)_{2n+1}} = \frac{1}{(-q;q)_\infty(a^2q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(a^2q^2;q^2)_n q^{n(n+3)/2}}{(q;q)_n}.$$

Upon multiplying both sides of this last identity by $aq(-q;q)_\infty(a^2q^2;q^2)_\infty$ and noting that $(-q;q)_\infty = 1/(q;q^2)_\infty$, we obtain, after replacing q by $-q$ and replacing n by $n-1$ on the right-hand side,

$$T_2 = -a \sum_{n=1}^{\infty} \frac{(a^2q^2;q^2)_{n-1}(-q)^{n(n+1)/2}}{(-q;-q)_{n-1}}. \tag{1.4.4}$$

If we substitute (1.4.3) and (1.4.4) into (1.4.2), we obtain our desired identity to complete the proof. □

Entry 1.4.7 (p. 16). *If a is any complex number, then*

$$\frac{(-aq;q)_\infty}{(-q;q)_\infty} \sum_{n=0}^{\infty} \frac{(aq;q)_n q^{n(n+1)}}{(q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{(a^2q^2;q^2)_n(-q)^{n(n+1)/2}}{(-q;-q)_n}$$
$$+ a \sum_{n=0}^{\infty} \frac{(-1)^n(a^2q^2;q^2)_{2n} q^{2n^2+4n+1}}{(q^4;q^4)_n}.$$

Proof. In (1.2.5), let $t = -q^2/a$ and $c = 0$. After letting $a \to \infty$, set $b = aq$. Multiplying both sides of the resulting identity by $(-aq;q)_\infty/(-q;q)_\infty$, we find that

$$\begin{aligned}\frac{(-aq;q)_\infty}{(-q;q)_\infty} \sum_{n=0}^{\infty} \frac{(aq;q)_n q^{n(n+1)}}{(q^2;q^2)_n} &= \frac{(a^2q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(aq)^{2n}}{(q;q)_{2n}(-q^2;q^2)_n} \\ &\quad + \frac{(a^2q^2;q^2)_\infty}{(-q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(aq)^{2n+1}}{(q;q)_{2n+1}(-q;q^2)_{n+1}} \\ &=: S_1 + S_2. \end{aligned} \tag{1.4.5}$$

Now in (1.2.1) with $h = 2$, set $a = 0$ and $c = -q$, and let b tend to 0. Then set $t = a^2q^2$. The result, after replacing q by $-q$ and simplifying, is given by

$$\sum_{n=0}^{\infty} \frac{(a^2q^2;q^2)_n(-q)^{n(n+1)/2}}{(-q;-q)_n} = \frac{(a^2q^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(aq)^{2n}}{(q;q)_{2n}(-q^2;q^2)_n} = S_1. \tag{1.4.6}$$

Next, in (1.2.1), set $h = 2$, replace q by q^2, and then set $b = a^2q^2$, $t = q^6/a$, and $c = 0$. After letting $a \to \infty$ and substantially simplifying, we find that

$$
\begin{aligned}
a\sum_{n=0}^{\infty}&\frac{(-1)^n(a^2q^2;q^2)_{2n}q^{2n^2+4n+1}}{(q^4;q^4)_n}\\
&=\frac{(a^2q^2;q^2)_\infty}{(-q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(aq)^{2n+1}}{(q;q)_{2n+1}(-q;q^2)_{n+1}}=S_2. \qquad (1.4.7)
\end{aligned}
$$

If we substitute (1.4.7) and (1.4.6) into (1.4.5), we obtain the desired identity for this entry. □

Entry 1.4.8 (p. 16). *For arbitrary complex numbers a and b,*

$$
\begin{aligned}
\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(aq;q)_n b^n q^{n^2}}{(q^2;q^2)_n}&=(-bq;q^2)_\infty\sum_{n=0}^{\infty}\frac{(aq)^{2n}}{(q;q)_{2n}(-bq;q^2)_n}\\
&\quad+(-bq^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{(aq)^{2n+1}}{(q;q)_{2n+1}(-bq^2;q^2)_n}.
\end{aligned}
$$

Proof. This entry is a further special case of (1.2.5); replace a by $-bq/t$, set $c=0$ and $b=aq$, and let $t\to 0$. □

In her thesis [225], Padmavathamma also proved Entry 1.4.8. For a combinatorial proof of Entry 1.4.8, see the paper by Berndt, Kim, and Yee [73].

The next entry is the first of several identities in this chapter that provide representations of theta functions or quotients of theta functions by basic hypergeometric series. We therefore review here Ramanujan's notations for theta functions and some basic facts about theta functions.

Recall that the Jacobi triple product identity [18, p. 21, Theorem 2.8], [54, p. 35, Entry 19] is given, for $|ab|<1$, by

$$
f(a,b):=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}=(-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \qquad (1.4.8)
$$

Deducible from (1.4.8) are the product representations of the classical theta functions [18, p. 23, Corollary 2.10], [54, pp. 36–37, Entry 22, equation (22.4)],

$$
\varphi(-q):=f(-q,-q)=\sum_{n=-\infty}^{\infty}(-1)^nq^{n^2}=\frac{(q)_\infty}{(-q)_\infty}, \qquad (1.4.9)
$$

$$
\psi(q):=f(q,q^3)=\sum_{n=0}^{\infty}q^{n(n+1)/2}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \qquad (1.4.10)
$$

$$
f(-q):=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^nq^{n(3n-1)/2}=(q;q)_\infty, \qquad (1.4.11)
$$

where we have employed the notation used by Ramanujan throughout his notebooks. The last equality in (1.4.11) is known as Euler's pentagonal number theorem. We also need the elementary result [54, p. 34, Entry 18(iii)]

$$f(-1, a) = 0, \tag{1.4.12}$$

for any complex number a with $|a| < 1$. Later, we need the fundamental property [54, p. 34]: For $|ab| < 1$ and each integer n,

$$f(a, b) = a^{n(n+1)/2} b^{n(n-1)/2} f\left(a(ab)^n, b(ab)^{-n}\right). \tag{1.4.13}$$

Entry 1.4.9 (p. 10). *Let $\varphi(-q)$ be defined by* (1.4.9) *above. Then*

$$\varphi(-q) \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n^2} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q^2;q^2)_n}. \tag{1.4.14}$$

First Proof of Entry 1.4.9. In (1.2.1), we set $h = 1$, $a = -q/\tau$, $b = \tau$, $c = q$, and $t = \tau$. Letting τ tend to 0, we find that

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n^2} = \frac{(-q)_\infty}{(q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q)_n(-q)_n}. \tag{1.4.15}$$

The desired result follows once we invoke the well-known product representation for $\varphi(-q)$ in (1.4.9). □

Second Proof of Entry 1.4.9. Our second proof is taken from the paper [73] by Berndt, Kim, and Yee.

Multiplying both sides of (1.4.15) by $(q)_\infty$, we obtain the equivalent identity

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n}(q^{n+1};q)_\infty = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q)_n}(-q^{n+1};q)_\infty, \tag{1.4.16}$$

since $(q^2;q^2)_\infty = (-q;q)_\infty(q;q)_\infty$. The left side of (1.4.16) is a generating function for the pair of partitions (π,ν), where π is a partition into n distinct parts and ν is a partition into distinct parts that are strictly larger than n and where the exponent of (-1) is the number of parts in ν. For a given partition pair (π, ν) generated by the left side of (1.4.16), let k be the number of parts in ν. Detach n from the each part of ν and attach k to each part of π. Then we obtain partition pairs (σ, λ), such that σ is a partition into k distinct parts and λ is a partition into distinct parts that are strictly larger than k, and the exponent of (-1) is the number of parts in σ. These partitions are generated by the right side of (1.4.16). Since this process is easily reversible, our proof is complete. □

The series on the left-hand sides of (1.4.14) and (1.4.18) below are the generating functions for the enumeration of gradual stacks with summits and stacks with summits, respectively [23]. Another generating function for gradual stacks with summits was found by Watson [279, p. 59], [75, p. 328], who showed that

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n^2} = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(q^2;q^2)_n}, \tag{1.4.17}$$

which is implicit in the work of Ramanujan in his lost notebook [244]. An elegant generalization of the concept of gradual stacks with summits has been devised by Yee, with her generating function generalizing that on the right-hand side of (1.4.17) [286, Theorem 5.2]. See Entry 6.3.1 for a significant generalization of Entry 1.4.10 involving two additional parameters.

Entry 1.4.10 (p. 10).

$$\sum_{n=0}^{\infty} \frac{q^n}{(q)_n^2} = \frac{1}{(q)_\infty^2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}. \tag{1.4.18}$$

Proof. In (1.2.1), set $h = 1$, $t = c = q$, and $a = 0$, and then let $b \to 0$. Entry 1.4.10 follows immediately. □

Entry 1.4.11 (p. 10).

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(q)_n^2} = \frac{1}{(q)_\infty^2} \left(1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n(n+1)/2}\right). \tag{1.4.19}$$

Proof. In (1.2.1), set $h = 1$, $a = 0$, $c = q$, and $t = q^2$. Now let $b \to 0$ to deduce that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{2n}}{(q)_n^2} &= \frac{1}{(q)_\infty^2}(1-q) \sum_{n=0}^{\infty} (-1)^n \frac{1-q^{n+1}}{1-q} q^{n(n+1)/2} \\
&= \frac{1}{(q)_\infty^2} \left(\sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} + \sum_{n=0}^{\infty} (-1)^{n+1} q^{(n+1)(n+2)/2} \right) \\
&= \frac{1}{(q)_\infty^2} \left(1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n(n+1)/2}\right).
\end{aligned}$$

□

Observe that the sum on the right sides in Entries 1.4.10 and 1.4.11 is a false theta function in the sense of L.J. Rogers. Several other entries in the lost notebook involve this false theta function; see [31, pp. 227–232] for some of these entries. In providing a combinatorial proof of Entry 1.4.11, Kim [189] was led to a generalization for which he supplied a combinatorial proof.

The following entry has been combinatorially proved by Berndt, Kim, and Yee [73].

Entry 1.4.12 (p. 10). *For $|a|, |b| < 1$ and any positive integer n,*

$$(-bq^n;q^n)_\infty \sum_{m=0}^{\infty} \frac{a^m q^{m(m+1)/2}}{(q;q)_m(-bq^n;q^n)_m} = (-aq;q)_\infty \sum_{m=0}^{\infty} \frac{b^m q^{nm(m+1)/2}}{(q^n;q^n)_m(-aq;q)_{nm}}. \tag{1.4.20}$$

Proof. In (1.2.1), set $h = n$ and let b tend to 0. Then set $t = -bq^n/a$ and let a tend to ∞. Finally, replace c by $-aq$. $\square$

Entry 1.4.13 (p. 11). *For* $|a| < 1$,

$$\sum_{n=0}^{\infty} \frac{(aq)^n}{(q^2;q^2)_n (bq;q)_n} = \frac{1}{(aq;q^2)_\infty (bq;q)_\infty} \sum_{n=0}^{\infty} \frac{(aq;q^2)_n b^{2n} q^{2n^2+n}}{(q;q)_{2n}} - \frac{1}{(aq^2;q^2)_\infty (bq;q)_\infty} \sum_{n=0}^{\infty} \frac{(aq^2;q^2)_n b^{2n+1} q^{2n^2+3n+1}}{(q;q)_{2n+1}}.$$

Proof. In (1.2.5), let $a = 0$ and let $b \to 0$. Then replace t by aq and c by bq. $\square$

In her doctoral dissertation [225], Padmavathamma gave another proof of Entry 1.4.13, and gave proofs of the following two entries as well.

Entry 1.4.14 (p. 11). *For any complex number* a,

$$(q^2;q^4)_\infty \sum_{n=0}^{\infty} \frac{(aq^2;q^2)_n q^{n(n+1)/2}}{(q;q)_n} = (aq^4;q^4)_\infty \sum_{n=0}^{\infty} \frac{(aq^2;q^4)_n q^{4n^2}}{(q^2;q^2)_{2n}} + (aq^2;q^4)_\infty \sum_{n=0}^{\infty} \frac{(aq^4;q^4)_n q^{4n^2+4n+1}}{(q^2;q^2)_{2n+1}}.$$

Proof. In Entry 1.4.13, replace q by q^2 and set $b = -1/q$. This yields

$$\sum_{n=0}^{\infty} \frac{a^n q^{2n}}{(q^4;q^4)_n (-q;q^2)_n} = \frac{1}{(aq^2;q^4)_\infty (-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(aq^2;q^4)_n q^{4n^2}}{(q^2;q^2)_{2n}} + \frac{1}{(aq^4;q^4)_\infty (-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(aq^4;q^4)_n q^{4n^2+4n+1}}{(q^2;q^2)_{2n+1}}.$$

Consequently, in order to prove the desired result, we must show that

$$(aq^2;q^4)_\infty (-q;q^2)_\infty (aq^4;q^4)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{2n}}{(q^4;q^4)_n (-q;q^2)_n} = (q^2;q^4)_\infty \sum_{n=0}^{\infty} \frac{(aq^2;q^2)_n q^{n(n+1)/2}}{(q;q)_n}, \tag{1.4.21}$$

and this follows from (1.2.1). More precisely, let $h = 2$, $c = -q$, and $a = 0$, and let b tend to 0. Then put $t = aq^2$ and simplify. $\square$

Entry 1.4.15 (p. 11). *If* a *is any complex number, then*

$$(q^2;q^4)_\infty \sum_{n=0}^{\infty} \frac{(aq^2;q^2)_n q^{(n+1)(n+2)/2}}{(q;q)_n} = (aq^4;q^4)_\infty \sum_{n=0}^{\infty} \frac{(aq^2;q^4)_n q^{4n^2+4n+1}}{(q^2;q^2)_{2n}} + (aq^2;q^4)_\infty \sum_{n=0}^{\infty} \frac{(aq^4;q^4)_n q^{4n^2+8n+4}}{(q^2;q^2)_{2n+1}}.$$

Proof. In Entry 1.4.13, replace q by q^2 and set $b = -q$. Upon multiplication of both sides by $q/(1+q)$, we find that

$$\sum_{n=0}^{\infty} \frac{a^n q^{2n+1}}{(q^4;q^4)_n(-q;q^2)_{n+1}} = \frac{1}{(aq^2;q^4)_\infty(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(aq^2;q^4)_n q^{4n^2+4n+1}}{(q^2;q^2)_{2n}} + \frac{1}{(aq^4;q^4)_\infty(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(aq^4;q^4)_n q^{4n^2+8n+4}}{(q^2;q^2)_{2n+1}}.$$

Consequently, in order to prove Entry 1.4.15, we must show that

$$(-q;q^2)_\infty(aq^2;q^4)_\infty(aq^4;q^4)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{2n+1}}{(q^4;q^4)_n(-q;q^2)_{n+1}} = (q^2;q^4)_\infty \sum_{n=0}^{\infty} \frac{(aq^2;q^2)_n q^{(n+1)(n+2)/2}}{(q;q)_n}. \tag{1.4.22}$$

This last identity follows from (1.2.1). Set $h = 2$, $c = -q^2$, and $a = 0$. Then let $b \to 0$. Setting $t = aq^2$ and multiplying both sides of the resulting identity by $q/(1+q)$, we complete the proof. □

Entry 1.4.16 (p. 11). *For any complex number a,*

$$(q;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n (aq;q)_{2n} q^{n(n+1)}}{(q^2;q^2)_n} = (aq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{(aq;q^2)_n q^{2n^2+n}}{(q;q)_{2n}} - (aq;q^2)_\infty \sum_{n=0}^{\infty} \frac{(aq^2;q^2)_n q^{2n^2+3n+1}}{(q;q)_{2n+1}}.$$

Proof. Set $b = 1$ in Entry 1.4.13 to deduce that

$$\sum_{n=0}^{\infty} \frac{(aq)^n}{(q^2;q^2)_n(q;q)_n} = \frac{1}{(aq;q^2)_\infty(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(aq;q^2)_n q^{2n^2+n}}{(q;q)_{2n}} - \frac{1}{(aq^2;q^2)_\infty(q;q)_\infty} \sum_{n=0}^{\infty} \frac{(aq^2;q^2)_n q^{2n^2+3n+1}}{(q;q)_{2n+1}}.$$

Therefore, in order to complete the proof of Entry 1.4.16, we must prove that

$$(q;q)_\infty(aq;q^2)_\infty(aq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{(aq)^n}{(q^2;q^2)_n(q;q)_n} = (q;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n (aq;q)_{2n} q^{n(n+1)}}{(q^2;q^2)_n}, \tag{1.4.23}$$

and this follows from (1.2.1) with $h = 2$, first setting $a = q^2/t$, then letting c and t tend to 0, and finally replacing b by aq. □

Entry 1.4.17 (p. 30). *For each positive integer n, the series*

$$(-aq)_\infty \sum_{m=0}^{\infty} \frac{b^m q^{m(m+1)/2}}{(q)_m(-aq)_{nm}}$$

is symmetric in a and b.

Proof. By an application of the q-binomial theorem (1.2.4),

$$\begin{aligned}(-aq)_\infty \sum_{m=0}^{\infty} \frac{b^m q^{m(m+1)/2}}{(q)_m(-aq)_{nm}} &= \sum_{m=0}^{\infty} \frac{b^m q^{m(m+1)/2}}{(q)_m}(-aq^{nm+1})_\infty \\ &= \sum_{m,j=0}^{\infty} \frac{b^m a^j q^{m(m+1)/2+j(j+1)/2+nmj}}{(q)_m(q)_j}.\end{aligned}$$

This last series is obviously symmetric in a and b, and so the proof is complete. ☐

Berndt, Kim, and Yee [73] found a combinatorial proof of Entry 1.4.17.

The next result from the top of page 27 of Ramanujan's lost notebook has lines drawn through it. Furthermore, the right-hand side has ellipses after the products forming the numerator and the denominator. If nothing is added, the result is clearly false. However, the following identity has the same left-hand side that Ramanujan gave, and the infinite products from the right-hand side of his proposed identity are isolated in front of our right-hand side.

Entry 1.4.18 (p. 27, corrected). *For any complex numbers a and b with $b \neq 0$,*

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(-a/b;q^2)_n b^n q^{n(n+1)/2}}{(q;q)_n(aq^2;q^2)_n} \\ &\quad = \frac{(-bq;q)_\infty}{(aq;q)_\infty}\left\{\frac{(-a/b;q^2)_\infty(aq;q)_\infty}{(aq^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-bq^2;q^2)_n}{(q^2;q^2)_n(-bq;q)_{2n}}\left(-\frac{a}{b}\right)^n\right\}. \qquad (1.4.24)\end{aligned}$$

Proof. In (1.2.1), take $h = 2$, and then replace a, c, and t by $-bq^2$, $-bq$, and $-a/b$, respectively. Now let $b \to 0$. Simplification then yields the desired result. ☐

1.5 Corollaries of (1.2.6) and (1.2.7)

The first two entries in this section were proved by G.N. Watson [278] and Andrews [7], with Berndt, Kim, and Yee [73] also providing a combinatorial proof of the former entry.

Entry 1.5.1 (p. 42). *If a is any complex number, then*

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n} = (-aq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q^2;q^2)_n(-aq^2;q^2)_n}$$
$$= (-aq;q^2)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2+n}}{(q^2;q^2)_n(-aq;q^2)_n}.$$

Proof. The first line follows by setting $t = -aq/b$ in (1.2.7) and letting $b \to 0$. The second line follows from the fact that each of the right-hand entries is equal to

$$\sum_{m,n=0}^{\infty} \frac{a^{m+n} q^{n^2+m^2+m+2mn}}{(q^2;q^2)_m(q^2;q^2)_n}.$$

To verify this last claim, first apply (1.2.4) to $(-aq^{2n+2};q^2)_\infty$. Secondly, apply (1.2.4) to $(-aq^{2n+1};q^2)_\infty$ and then switch the roles of m and n. □

M. Somos has observed that if we set

$$F(a,b;q) := (-bq;q^2)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q^2;q^2)_n(-bq;q^2)_n},$$

then

$$F(a,b;q) = F(b,a;q). \tag{1.5.1}$$

Entry 1.5.1 then follows by taking $b = aq$ in (1.5.1). To prove (1.5.1), return to Entry 1.4.12, set $n = 1$, and replace q by q^2, a by a/q, and b by b/q. Then we easily see that (1.4.20) reduces to (1.5.1).

Entry 1.5.2 (p. 42). *If a is any complex number, then*

$$\sum_{n=0}^{\infty} \frac{a^{2n} q^{4n^2}}{(q^4;q^4)_n} = (aq;q^2)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q^2;q^2)_n(aq;q^2)_n}.$$

Proof. Replace q by q^2 in (1.2.6). Then set $b = -aq/t$ and let $t \to 0$. □

Entry 1.5.3 (p. 26).

$$(q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q^2;q^2)_n}.$$

Proof. In Entry 1.5.2, replace q by $\sqrt{q}$, and then set $a = -\sqrt{q}$. Using Euler's identity, we find that the result simplifies to the equality above. □

Entry 1.5.4 (p. 26).

$$(q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n^2-n}}{(q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)/2}}{(q^2;q^2)_n}.$$

Proof. In (1.2.6), replace t by $1/b$ and let $b \to \infty$. Hence,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{2n^2-n}}{(q^2;q^2)_n} &= (-q;q)_\infty \left(2 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n(-q;q)_{n-1}} \right) \\
&= (-q;q)_\infty \left(2 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n-1)/2}(1-q^n+q^n)}{(q^2;q^2)_{n-1}(1-q^n)} \right) \\
&= (-q;q)_\infty \left(2 + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q^2;q^2)_{n-1}} \right. \\
&\qquad \left. + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}(1+q^n)}{(q^2;q^2)_n} \right) \\
&= (-q;q)_\infty \left(2 - \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q^2;q^2)_n} \right. \\
&\qquad \left. + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q^2;q^2)_n} + \sum_{n=1}^{\infty} \frac{(-1)^n q^{n(n+3)/2}}{(q^2;q^2)_n} \right) \\
&= (-q;q)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+3)/2}}{(q^2;q^2)_n}.
\end{aligned}$$

By Euler's identity, this last identity is equivalent to that of Entry 1.5.4. □

1.6 Corollaries of (1.2.8), (1.2.9), and (1.2.10)

Entry 1.6.1 (p. 362). *If a and b are any complex numbers, then*

$$(aq)_\infty \sum_{n=0}^{\infty} \frac{b^n q^{n^2}}{(q)_n (aq)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n (b/a)_n a^n q^{n(n+1)/2}}{(q)_n}.$$

Proof. Replace t by $t/(ab)$ in (1.2.9) and let a and b tend to ∞. This then yields the identity

$$\sum_{n=0}^{\infty} \frac{t^n q^{n^2-n}}{(q)_n (c)_n} = \frac{1}{(c)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (t/c)_n c^n q^{n(n-1)/2}}{(q)_n}. \tag{1.6.1}$$

Replacing t by bq and c by aq in (1.6.1), we complete the proof. □

Entry 1.6.1 is identical to Entry 9 in Chapter 16 of Ramanujan's second notebook [243], [54, p. 18]. Earlier proofs of Entry 1.6.1 were given by V. Ramamani [234] and by Ramamani and K. Venkatachaliengar [235]. L. Carlitz [99] posed the special case $a = -1$ of Entry 1.6.1 as a problem. S. Bhargava and C. Adiga [81] proved a generalization of Entry 1.6.1, while H.M. Srivastava [268] later established an equivalent formulation of their result. Lastly, Berndt, Kim, and Yee [73] have devised a bijective proof of Entry 1.6.1.

Entry 1.6.2 (p. 28). *For any complex number a,*

$$\sum_{n=0}^{\infty} a^n q^{n^2} = 1 + \sum_{n=1}^{\infty} \frac{(-q;q)_{n-1} a^n q^{n(n+1)/2}}{(-aq^2;q^2)_n}.$$

Proof. Subtracting 1 from both sides of this entry, shifting the summation indices down by 1 on each side, and dividing both sides by aq, we see that the identity above is equivalent to the identity

$$\sum_{n=0}^{\infty} a^n q^{n^2+2n} = \sum_{n=0}^{\infty} \frac{(-q;q)_n a^n q^{n(n+3)/2}}{(-aq^2;q^2)_{n+1}}.$$

Now in (1.2.8), replace a by $-aq^2/t$, set $b = q^2$, and let $t \to 0$. The resulting identity then simplifies to that of Entry 1.6.2. ∎

See the paper [73] by Berndt, Kim, and Yee for a combinatorial proof of Entry 1.6.2.

The next result does not properly belong under the heading of this section, but we have put it here because of its similarity to the previous entry. We note that the case $a = 1$ is Entry 9.3.2 of Part I [31, p. 229].

Entry 1.6.3 (p. 28). *For any complex number a,*

$$\sum_{n=0}^{\infty} (-1)^n a^n q^{n(n+1)/2} = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n a^{2n} q^{n(n+1)}}{(-aq;q)_{2n+1}}.$$

Proof. In (1.2.1), let $h = 2$, replace a by a^2q^2/t, then set $b = q$ and $c = -aq^2$, and let $t \to 0$. After simplification, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n a^{2n} q^{n(n+1)}}{(-aq;q)_{2n+1}} &= (q;q)_\infty (aq;q)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n (aq;q)_n} \\ &= \sum_{n=0}^{\infty} (-1)^n a^n q^{n(n+1)/2},\end{aligned}$$

where we applied (1.2.1) with $h = 1$ and replaced a, b, c, and t, respectively, by 0, 0, aq, and q. ∎

Entry 1.6.4 (p. 38). *For $|aq| < 1$,*

$$\sum_{n=0}^{\infty} \frac{(-aq)^n}{(-aq^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n+1)/2}}{(-aq;q)_n}. \tag{1.6.2}$$

Proof. In (1.2.8), set $a = 0$ and $b = q^2$, and then replace t by $-a$. Simplification yields

$$\sum_{n=0}^{\infty} \frac{(-aq)^n}{(-aq^2;q^2)_n} = (1+a)\sum_{n=0}^{\infty}(-q;q)_n(-a)^n$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n+1)/2}}{(-aq;q)_n},$$

where the last line follows from (1.2.9), wherein we replaced a by $-q$ and b by q, then set $t=-a$, and let $c \to 0$. □

Entry 1.6.4 can also be derived from Entry 9.2.6 in our first book on the lost notebook [31, p. 226]; this entry is on page 30 of [244]. In fact, when Berndt and A.J. Yee gave a combinatorial proof of Entry 9.2.6 in [78], after some elementary manipulation and the replacement of a by $-aq$ in Entry 9.2.6, they derived (1.6.2), for which they gave a bijective proof. Berndt, Kim, and Yee [73] have recently found a simpler bijective proof of (1.6.2).

Entry 1.6.5 (p. 38). *For any complex number a,*

$$\sum_{m=0}^{\infty} \frac{a^m q^{m(m+1)}}{(q^2;q^2)_m(1+aq^{2m+1})} = (-aq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-a)^n q^{n(n+1)/2}}{(-aq;q)_n}. \tag{1.6.3}$$

First Proof of Entry 1.6.5. Expanding $1/(1+aq^{2n+1})$ in a geometric series, inverting the order of summation, and using (1.2.4), we find that, for $|aq|<1$,

$$\sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)}}{(q^2;q^2)_n(1+aq^{2n+1})} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^m a^{n+m} q^{n(n+1)+m(2n+1)}}{(q^2;q^2)_n}$$
$$= \sum_{m=0}^{\infty}(-aq)^m \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1+2m)}}{(q^2;q^2)_n}$$
$$= \sum_{m=0}^{\infty}(-aq)^m(-aq^{2+2m};q^2)_\infty$$
$$= (-aq^2;q^2)_\infty \sum_{m=0}^{\infty} \frac{(-aq)^m}{(-aq^2;q^2)_m}$$
$$= (-aq^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-a)^n q^{n(n+1)/2}}{(-aq;q)_n},$$

where the last line follows from Entry 1.6.4. The desired result now follows by analytic continuation in a. □

Second Proof of Entry 1.6.5. Our second proof is taken from a paper by Berndt, Kim, and Yee [73].

By Entry 1.6.4, the identity (1.6.3) can be written in the equivalent form

$$\sum_{m=0}^{\infty} \frac{a^m q^{m(m+1)}}{(q^2;q^2)_m(1+aq^{2m+1})} = \sum_{n=0}^{\infty}(-aq)^n(-aq^{2n+2};q^2)_\infty.$$

Note that $(-aq^{2n+2};q^2)_\infty$ generates partitions into distinct even parts, each greater than or equal to $2n+2$, with the exponent of a denoting the number of parts. Let m be the number of parts generated by a partition arising from $(-aq^{2n+2};q^2)_\infty$. Detach $2n$ from each of the m parts. Combining this with $(-aq)^n$, we obtain $(-aq^{2m+1})^n$. However, note that, for $n \geq 0$, all of these odd parts are generated by $1/(1+aq^{2m+1})$, and each part is weighted by $-a$. The remaining parts, which are even, are generated by

$$\sum_{m=0}^{\infty} \frac{a^m q^{m(m+1)}}{(q^2;q^2)_m}.$$

For these partitions into m distinct even parts, the exponent of a again denotes the number of parts. □

Entry 1.6.6 (p. 35). *Recall that* $\psi(q)$ *is defined by* (1.4.10). *Then*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n}}{(q^2;q^2)_n(1-q^{2n+1})} = \psi(q).$$

Proof. Set $a=-1$ in Entry 1.6.5. Using (1.2.4), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+n}}{(q^2;q^2)_n(1-q^{2n+1})} &= (q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n} \\ &= (q^2;q^2)_\infty(-q;q)_\infty = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \psi(q),\end{aligned}$$

by Euler's identity and (1.4.10). □

Entry 1.6.7 (p. 40). *For* $|a|<1$,

$$(a)_\infty \sum_{n=0}^{\infty} \frac{a^n}{(q)_n(bq)_n} = \sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2}}{(q)_n(bq)_n}.$$

Proof. In (1.2.10), let both a and b tend to 0. Then replace t by a and c by bq, and lastly multiply both sides by $(a)_\infty$. □

1.7 Corollaries of Section 1.2 and Auxiliary Results

Up to now in this chapter, we have concentrated on results from the lost notebook that can be traced to pairs of antecedent formulas proved in Section 1.2. In this section, we also often draw on several of the results from Section 1.2, but additionally we require other formulas that have appeared often in Ramanujan's work.

The Rogers–Fine identity [149, p. 15]

$$\sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} \tau^n = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\alpha\tau q/\beta)_n \beta^n \tau^n q^{n^2-n}(1-\alpha\tau q^{2n})}{(\beta)_n (\tau)_{n+1}} \tag{1.7.1}$$

is needed in this chapter. Ramanujan frequently used this identity in the lost notebook; see Chapter 9 of [31], which is entirely devoted to formulas in the lost notebook derived from (1.7.1).

The next two results are, in fact, special cases of the q-binomial theorem, (1.2.2). However, it will be more convenient to invoke them using the q-binomial coefficients, which are defined by

$$\begin{bmatrix} k \\ \ell \end{bmatrix}_q = \begin{bmatrix} k \\ \ell \end{bmatrix} := \begin{cases} 0, & \text{if } \ell < 0 \text{ or } \ell > k, \\ \dfrac{(q)_k}{(q)_\ell (q)_{k-\ell}}, & \text{otherwise.} \end{cases} \tag{1.7.2}$$

For any complex numbers a, b, and any nonnegative integer n,

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^j a^{-j} b^j q^{j(j-1)/2} = (b/a)_n. \tag{1.7.3}$$

Also, for $|z| < 1$ and any nonnegative integer N,

$$\sum_{n=0}^{\infty} \begin{bmatrix} n+N \\ n \end{bmatrix} z^n = \frac{1}{(z)_{N+1}}. \tag{1.7.4}$$

Entry 1.7.1 (p. 5). *For any complex number a,*

$$\sum_{n=0}^{\infty} \frac{(-a;q)_{2n+1} q^{2n+1}}{(q;q^2)_{n+1}} + \sum_{n=0}^{\infty} (-a)^n q^{n(n+1)/2} = \frac{(-aq;q)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-a)^n q^{n(n+1)/2}}{(-aq;q)_n}.$$

Proof. In (1.2.5), set $t = q$, $b = -q$, and $a = 0$; then replace c by aq. We therefore deduce that

$$\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n (aq;q)_n} = \frac{(-q;q)_\infty}{(aq;q)_\infty (q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-a;q)_{2n} q^{2n}}{(q^2;q^2)_n} - \frac{(-q;q)_\infty}{(aq;q)_\infty (q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-a;q)_{2n+1} q^{2n+1}}{(q;q^2)_{n+1}}. \tag{1.7.5}$$

In (1.2.1), set $h = 1$, $a = 0$, and $t = q$. Then, replacing c by aq and letting b tend to 0, we find that

$$\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_n (aq;q)_n} = \frac{1}{(q;q)_\infty (aq;q)_\infty} \sum_{n=0}^{\infty} (-a)^n q^{n(n+1)/2}. \tag{1.7.6}$$

Equating the right-hand sides of (1.7.6) and (1.7.5), and multiplying the result by $(q;q)_\infty(aq;q)_\infty$, we deduce that

$$\sum_{n=0}^{\infty}(-a)^n q^{n(n+1)/2} = -\sum_{n=0}^{\infty}\frac{(-a;q)_{2n+1}q^{2n+1}}{(q;q^2)_{n+1}} + \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-a;q)_{2n}q^{2n}}{(q^2;q^2)_n}.$$

We therefore will be finished with the proof if we can show that

$$\sum_{n=0}^{\infty}\frac{(-a;q)_{2n}q^{2n}}{(q^2;q^2)_n} = \frac{(-aq;q)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-a)^n q^{n(n+1)/2}}{(-aq;q)_n}. \tag{1.7.7}$$

To that end, we apply (1.2.1) with $h=1$, q replaced by q^2, $c=0$, $t=q^2$, a replaced by $-a$, and b replaced by $-aq$ to find that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-a;q)_{2n}q^{2n}}{(q^2;q^2)_n} &= \frac{(-aq;q^2)_\infty(-aq^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-aq)^n}{(-aq^2;q^2)_n}\\ &= \frac{(-aq;q)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-a)^n q^{n(n+1)/2}}{(-aq;q)_n},\end{aligned}$$

by Entry 1.6.4. Thus, (1.7.7) has been proved, and so the proof of Entry 1.7.1 is complete. □

Entry 1.7.2 (p. 5). *If $|b|<1$ and a is an arbitrary complex number, then*

$$\sum_{n=0}^{\infty}\frac{(-1)^n(-q;q)_n(-aq/b;q)_n b^n}{(aq;q^2)_{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n(-aq/b;q)_n b^n q^{n(n+1)/2}}{(-b;q)_{n+1}}.$$

Proof. In (1.2.1), set $h=2$ and $t=q^2$, and replace b, c, and a by $-b$, aq, and aq, respectively. Consequently,

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(-1)^n(-q;q)_n(-aq/b;q)_n b^n}{(aq;q^2)_{n+1}}\\ &\qquad= \frac{(aq^2;q^2)_\infty(q^2;q^2)_\infty}{(-b;q)_\infty}\sum_{n=0}^{\infty}\frac{(-b;q^2)_n(-bq;q^2)_n}{(q^2;q^2)_n(aq^2;q^2)_n}q^{2n}\\ &\qquad= \sum_{n=0}^{\infty}\frac{(-aq/b;q^2)_n}{(-b;q^2)_{n+1}}(-bq)^n,\end{aligned}$$

where the last equality follows from (1.2.1) with $h=1$, q replaced by q^2, $t=q^2$, and a, b, and c replaced by $-b$, $-bq$, and aq^2, respectively.

To this last expression we apply the Rogers–Fine identity (1.7.1) with q replaced by q^2, $\alpha = -aq/b$, $\beta = -bq^2$, and $\tau = -bq$ to deduce that, after multiplying both sides by $1/(1+b)$,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-aq/b;q^2)_n}{(-b;q^2)_{n+1}}(-bq)^n \\
&\quad= \sum_{n=0}^{\infty} \frac{(-aq/b;q^2)_n(-aq^2/b;q^2)_n(-bq^2)^n(-bq)^n q^{2n^2-2n}(1-aq^{4n+2})}{(-b;q^2)_{n+1}(-bq;q^2)_{n+1}} \\
&\quad= \sum_{n=0}^{\infty} \frac{(-aq/b;q^2)_n(-aq^2/b;q^2)_n b^{2n} q^{2n^2+n}(1-aq^{4n+2})}{(-b;q^2)_{n+1}(-bq;q^2)_{n+1}} \\
&\quad= \sum_{n=0}^{\infty} \frac{(-aq/b;q)_{2n} b^{2n} q^{2n^2+n}(1-aq^{4n+2})}{(-b;q)_{2n+2}} \\
&\quad= \sum_{n=0}^{\infty} \frac{(-aq/b;q)_{2n} b^{2n} q^{2n^2+n}}{(-b;q)_{2n+1}}\left(1+\frac{-bq^{2n+1}-aq^{4n+2}}{1+bq^{2n+1}}\right) \\
&\quad= \sum_{n=0}^{\infty} \frac{(-aq/b;q)_{2n} b^{2n} q^{2n^2+n}}{(-b;q)_{2n+1}} - \sum_{n=0}^{\infty} \frac{(-aq/b;q)_{2n+1} b^{2n+1} q^{(n+1)(2n+1)}}{(-b;q)_{2n+2}} \\
&\quad= \sum_{n=0}^{\infty} \frac{(-1)^n(-aq/b;q)_n b^n q^{n(n+1)/2}}{(-b;q)_{n+1}},
\end{aligned}$$

which is the desired result. □

Entry 1.7.3 (p. 30). *For any complex numbers a and c,*

$$\sum_{n=0}^{\infty} \frac{(c/a;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(aq;q)_n} = \frac{(cq;q^2)_\infty}{(aq;q)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n(a^2q/c;q^2)_n c^n q^{n^2+n}}{(q^2;q^2)_n(cq;q^2)_n}.$$

Proof. First, applying (1.2.9) with a, b, c, and t replaced by $-q/\tau$, c/a, aq, and $a\tau$, respectively, and letting τ tend to 0, we find that

$$\sum_{n=0}^{\infty} \frac{(c/a;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(aq;q)_n} = \frac{(a^2q/c;q)_\infty}{(aq;q)_\infty} \sum_{n=0}^{\infty} \frac{(c^2/a^2;q^2)_n}{(q;q)_n}\left(\frac{a^2q}{c}\right)^n. \tag{1.7.8}$$

Second, we invoke (1.2.8) with a, b, and t replaced by 0, c^2/a^2, and a^2q/c, respectively, to deduce that

$$\sum_{n=0}^{\infty} \frac{(c^2/a^2;q^2)_n}{(q;q)_n}\left(\frac{a^2q}{c}\right)^n = \frac{(cq;q^2)_\infty}{(a^2q/c;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(c^2/a^2;q^2)_n}{(q^2;q^2)_n(cq;q^2)_n}\left(\frac{a^2q^2}{c}\right)^n. \tag{1.7.9}$$

Combining (1.7.8) and (1.7.9), we see that

$$
\begin{aligned}
\sum_{n=0}^{\infty} & \frac{(c/a;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(aq;q)_n} \\
&= \frac{(a^2q^2/c;q^2)_\infty(cq;q^2)_\infty}{(aq;q)_\infty} \sum_{n=0}^{\infty} \frac{(c^2/a^2;q^2)_n}{(q^2;q^2)_n(cq;q^2)_n}\left(\frac{a^2q^2}{c}\right)^n \\
&= \frac{(a^2q^2/c;q^2)_\infty(cq;q^2)_\infty}{(aq;q)_\infty}\frac{(a^2q/c;q^2)_\infty(cq^2;q^2)_\infty}{(a^2q^2/c;q^2)_\infty(cq;q^2)_\infty} \\
&\quad \times \sum_{n=0}^{\infty} \frac{(c^2/a^2;q^2)_n}{(q^2;q^2)_n(cq^2;q^2)_n}\left(\frac{a^2q}{c}\right)^n, \qquad (1.7.10)
\end{aligned}
$$

where in the last line we applied (1.2.9) with q replaced by q^2 and with a, b, c, and t replaced respectively by 0, c^2/a^2, cq, and a^2q^2/c.

On the other hand, by (1.2.1) with q replaced by q^2, $h=1$, and a, b, c, and t replaced by q^2/τ, a^2q/c, cq, and $c\tau$, we find that, upon letting $\tau \to 0$,

$$
\begin{aligned}
\sum_{n=0}^{\infty} & \frac{(-1)^n(a^2q/c;q^2)_n c^n q^{n^2+n}}{(q^2;q^2)_n(cq;q^2)_n} \\
&= \frac{(a^2q/c;q^2)_\infty(cq^2;q^2)_\infty}{(cq;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(c^2/a^2;q^2)_n}{(q^2;q^2)_n(cq^2;q^2)_n}\left(\frac{a^2q}{c}\right)^n. \qquad (1.7.11)
\end{aligned}
$$

Equating the left-hand side of (1.7.11) multiplied by $(cq;q^2)_\infty/(aq;q)_\infty$ with the left-hand side of (1.7.10), we arrive at

$$
\begin{aligned}
\frac{(cq;q^2)_\infty}{(aq;q)_\infty} & \sum_{n=0}^{\infty} \frac{(-1)^n(a^2q/c;q^2)_n c^n q^{n^2+n}}{(q^2;q^2)_n(cq;q^2)_n} \\
&= \frac{(a^2q/c;q^2)_\infty(cq^2;q^2)_\infty}{(aq;q)_\infty} \sum_{n=0}^{\infty} \frac{(c^2/a^2;q^2)_n}{(q^2;q^2)_n(cq^2;q^2)_n}\left(\frac{a^2q}{c}\right)^n \\
&= \sum_{n=0}^{\infty} \frac{(c/a;q)_n a^n q^{n(n+1)/2}}{(q;q)_n(aq;q)_n},
\end{aligned}
$$

which is what we wanted to prove. □

The next two entries are, respectively, the cases $a=i$, $b=-i$ and $a=iq$, $b=-iq$ in Andrews's paper on the q-analogue of Kummer's theorem [15, p. 526, equation (1.8)].

Entry 1.7.4 (p. 34). *Recalling that* $\varphi(-q)$ *is defined in* (1.4.9), *we have*

$$
\sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^{n(n+1)/2}}{(q;q)_n(q;q^2)_n} = \frac{\varphi(-q^4)}{\varphi(-q)}.
$$

Proof. In (1.2.1), set $h = 2$ and $c = a = -q$, and let $b \to 0$ to deduce that

$$\sum_{n=0}^{\infty} \frac{(t;q^2)_n q^{n(n+1)/2}}{(q;q)_n(-tq;q^2)_n} = \frac{(-q;q)_\infty (t;q^2)_\infty}{(-tq;q^2)_\infty} \sum_{n=0}^{\infty} \frac{t^n}{(q^4;q^4)_n}$$
$$= \frac{(-q;q)_\infty (tq^2;q^2)_\infty}{(-tq;q^2)_\infty (tq^4;q^4)_\infty}, \qquad (1.7.12)$$

by (1.2.3) with $b = t$ and q replaced by q^4. If we now set $t = -1$ above, the left-hand side of (1.7.12) reduces to the left-hand side in Entry 1.7.4, while the right-hand side reduces to

$$\frac{(-q;q)_\infty(-q^2;q^2)_\infty}{(q;q^2)_\infty(-q^4;q^4)_\infty} = \frac{(-q;q)_\infty(-q^2;q^2)_\infty(q^2;q^2)_\infty}{(q;q)_\infty(-q^4;q^4)_\infty}$$
$$= \frac{(-q;q)_\infty(q^4;q^4)_\infty}{(q;q)_\infty(-q^4;q^4)_\infty} = \frac{\varphi(-q^4)}{\varphi(-q)},$$

by (1.4.9), as desired. □

Entry 1.7.5 (p. 35). *If* $\varphi(-q)$ *and* $\psi(q)$ *are defined by* (1.4.9) *and* (1.4.10), *respectively, then*

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n(n+1)/2}}{(q;q)_n(q;q^2)_{n+1}} = \frac{\psi(-q^2)}{\varphi(-q)}.$$

Proof. In (1.2.1), take $h = 2$, $a = c = -q$, $b = 0$, and $t = -q^2$, and multiply both the numerator and the denominator of the resulting identity by $1 - q$. Then use (1.2.3) with $b = -q^2$ and q replaced by q^4. Accordingly, after using Euler's identity, we find that

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n(n+1)/2}}{(q;q)_n(q;q^2)_{n+1}} = (-q;q)_\infty^2(-q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-q^2)^n}{(q^4;q^4)_n}$$
$$= \frac{(-q;q)_\infty^2(-q^2;q^2)_\infty}{(-q^2;q^4)_\infty}$$
$$= \frac{(-q;q)_\infty(-q^2;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(q^2;q^2)_\infty(-q^2;q^4)_\infty}$$
$$= \frac{(-q;q)_\infty(q^4;q^4)_\infty}{(q;q)_\infty(-q^2;q^4)_\infty} = \frac{\psi(-q^2)}{\varphi(-q)},$$

by (1.4.9) and (1.4.10). This completes the proof. □

The next entry can be found in Slater's compendium [262, equation (35)].

Entry 1.7.6 (p. 35). *Recall that* $f(a,b)$ *is defined in* (1.4.8). *Then*

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{(n+1)(n+2)/2}}{(q;q)_n(q;q^2)_{n+1}} = \frac{qf(-q,-q^7)}{\varphi(-q)}.$$

Proof. In (1.2.1), set $h = 2$, $a = c = -q^2$, $t = -q$, and $b = 0$. Multiplying both sides of the resulting identity by $q/(1+q)$, we find that

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{(n+1)(n+2)/2}}{(q;q)_n (q;q^2)_{n+1}} = \frac{q(-q;q^2)_\infty}{(q;q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{(-q)^n}{(-q;-q)_{2n+1}}. \tag{1.7.13}$$

Using (1.2.3) twice, with $b = \sqrt{q}$ and $b = -\sqrt{q}$, respectively, using Euler's identity, and employing the Jacobi triple product identity (1.4.8) three times altogether, we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^n}{(q;q)_{2n+1}} &= \frac{1}{2} \sum_{n=0}^{\infty} (1-(-1)^n) \frac{q^{(n-1)/2}}{(q;q)_n} \\
&= \frac{1}{2\sqrt{q}} \left(\frac{1}{(\sqrt{q};q)_\infty} - \frac{1}{(-\sqrt{q};q)_\infty} \right) \\
&= \frac{1}{2\sqrt{q}(q;q)_\infty} \Big((-q^{1/2};q^2)_\infty (-q^{3/2};q^2)_\infty (q^2;q^2)_\infty \\
&\quad - (q^{1/2};q^2)_\infty (q^{3/2};q^2)_\infty (q^2;q^2)_\infty \Big) \\
&= \frac{1}{2\sqrt{q}(q;q)_\infty} \left(\sum_{n=-\infty}^{\infty} q^{n^2-n/2} - \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2-n/2} \right) \\
&= \frac{1}{\sqrt{q}(q;q)_\infty} \sum_{n=-\infty}^{\infty} q^{(2n+1)^2-(2n+1)/2} \\
&= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} q^{4n^2+3n} \\
&= \frac{f(q,q^7)}{(q;q)_\infty}.
\end{aligned} \tag{1.7.14}$$

Replacing q by $-q$ in (1.7.14), substituting the result in (1.7.13), and using Euler's identity, we deduce that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{(n+1)(n+2)/2}}{(q;q)_n (q;q^2)_{n+1}} &= \frac{q(-q;q^2)_\infty f(-q,-q^7)}{(q;q^2)_\infty^2 (q^2;q^2)_\infty (-q;q^2)_\infty} \\
&= \frac{q(-q;q)_\infty f(-q,-q^7)}{(q;q)_\infty} \\
&= \frac{qf(-q,-q^7)}{\varphi(-q)},
\end{aligned}$$

which is the desired result. □

Entry 1.7.7 (p. 35). *If $f(a,b)$ is defined by* (1.4.8), *then*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{(n+1)(n+2)/2}}{(q)_n (1-q^{2n+1})} = qf(q,q^7).$$

Proof. Expanding $1/(1-q^{2n+1})$ in a geometric series, inverting the order of summation, appealing to (1.2.4), and using (1.7.14), we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n q^{(n+1)(n+2)/2}}{(q)_n(1-q^{2n+1})} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^n q^{(n+1)(n+2)/2+m(2n+1)}}{(q)_n}\\ &= \sum_{m=0}^{\infty} q^{m+1}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n-1)/2+n(2m+2)}}{(q)_n}\\ &= \sum_{m=0}^{\infty} q^{m+1}(q^{2m+2})_\infty\\ &= q(q)_\infty\sum_{m=0}^{\infty}\frac{q^m}{(q)_{2m+1}}\\ &= qf(q,q^7),\end{aligned}$$

which completes the proof. □

Entry 1.7.8 can be found in Slater's paper [262, equation (37)].

Entry 1.7.8 (p. 35). *If* $f(a,b)$ *is defined by* (1.4.8), *then*

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n(n+1)/2}}{(q;q)_n(q;q^2)_{n+1}} = \frac{f(-q^3,-q^5)}{\varphi(-q)}. \tag{1.7.15}$$

Proof. In (1.2.1), we set $h=2$, $a=-q^2$, $b=0$, and $c=t=-q$. Upon using Euler's identity, we find that

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n(n+1)/2}}{(q;q)_n(q;q^2)_{n+1}} = (-q;q)_\infty^2(-q;q^2)_\infty\sum_{n=0}^{\infty}\frac{(-q)^n}{(-q;-q)_{2n}}. \tag{1.7.16}$$

We now proceed as we did in the proof of Entry 1.7.6. We apply (1.2.3) twice, with $b=\sqrt{q},-\sqrt{q}$, and use the Jacobi triple product identity (1.4.8). Accordingly,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^n}{(q;q)_{2n}} &= \frac{1}{2}\sum_{n=0}^{\infty}(1+(-1)^n)\frac{q^{n/2}}{(q;q)_n}\\ &= \frac{1}{2}\left(\frac{1}{(\sqrt{q};q)_\infty}+\frac{1}{(-\sqrt{q};q)_\infty}\right)\\ &= \frac{1}{2(q;q)_\infty}\Big((-q^{1/2};q^2)_\infty(-q^{3/2};q^2)_\infty(q^2;q^2)_\infty\\ &\qquad +(q^{1/2};q^2)_\infty(q^{3/2};q^2)_\infty(q^2;q^2)_\infty\Big)\\ &= \frac{1}{2(q;q)_\infty}\left(\sum_{n=-\infty}^{\infty} q^{n^2+n/2}+\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2+n/2}\right)\end{aligned}$$

$$= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} q^{4n^2+n}$$
$$= \frac{f(q^3,q^5)}{(q;q)_\infty}. \tag{1.7.17}$$

Replacing q by $-q$ in (1.7.17), combining the result with (1.7.16), and using Euler's identity, we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(n+1)/2}}{(q;q)_n(q;q^2)_{n+1}} &= \frac{(-q;q)_\infty^2(-q;q^2)_\infty f(-q^3,-q^5)}{(q^2;q^2)_\infty(-q;q^2)_\infty} \\ &= \frac{(-q;q)_\infty f(-q^3,-q^5)}{(q;q)_\infty} \\ &= \frac{f(-q^3,-q^5)}{\varphi(-q)},\end{aligned}$$

by (1.4.9). This therefore completes the proof. □

Entry 1.7.9 (p. 35). *With $f(a,b)$ defined by* (1.4.8), *we have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q)_n(1-q^{2n+1})} = f(q^3,q^5).$$

Proof. Employing the q-binomial theorem (1.2.4) and proceeding as we did in (1.7.17), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q)_n(1-q^{2n+1})} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2+m(2n+1)}}{(q)_n} \\ &= \sum_{m=0}^{\infty} q^m (q^{2m+1})_\infty \\ &= (q)_\infty \sum_{m=0}^{\infty} \frac{q^m}{(q)_{2m}} \\ &= \frac{1}{2}(q)_\infty \sum_{m=0}^{\infty} (1+(-1)^m)\frac{q^{m/2}}{(q)_m} \\ &= \frac{1}{2}(q)_\infty \left(\frac{1}{(q^{1/2})_\infty} + \frac{1}{(-q^{1/2})_\infty}\right) \\ &= \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} q^{n^2+n/2} + \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+n/2}\right) \\ &= \sum_{n=-\infty}^{\infty} q^{4n^2+n} = f(q^3,q^5),\end{aligned}$$

by (1.4.8). □

Entry 1.7.10 (p. 3). *Recall that* $\psi(q)$ *is defined in* (1.4.10). *Then*

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(q^2;q^2)_n^2} = \frac{1}{\psi(q)}. \tag{1.7.18}$$

Proof. We utilize the q-Gauss summation theorem (1.3.1). Replace q by q^2. Then set $b = q$ and $c = q^2$. Letting $a \to \infty$, we find that

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(q^2;q^2)_n^2} = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} = \frac{1}{\psi(q)},$$

by (1.4.10). □

Entry 1.7.10 is identical to Entry 4.2.6. I. Pak [226] asked for a combinatorial proof of (1.7.18), but there is a misprint in his formulation. Berndt, Kim, and Yee [73] observed that if in Entry 1.3.2 we replace q by q^2 and then set $b = 1$ and $a = 1/q$, we obtain Entry 1.7.10 with q replaced by $-q$. Since these authors also gave a combinatorial proof of Entry 1.3.2, this gives the desired combinatorial proof sought by Pak.

Entry 1.7.11 (p. 41). *With* $f(a, b)$ *defined by* (1.4.8) *and* $\psi(q)$ *defined by* (1.4.10),

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(q^2;q^2)_n} = \frac{\psi(q^4)}{f(q, q^7)}.$$

Proof. In (1.2.1), let $h = 1$, replace q by q^2, then set $a = q/t$, $b = q$, and $c = 0$, and lastly let $t \to 0$ and simplify. Using (1.7.17), (1.4.8) twice, and (1.4.10), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(q^2;q^2)_n} &= (q;q^2)_\infty^2 \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_{2n}} \\
&= \frac{(q;q^2)_\infty^2}{(q;q)_\infty} f(q^3, q^5) \\
&= \frac{(q;q^2)_\infty (-q^3;q^8)_\infty (-q^5;q^8)_\infty (q^8;q^8)_\infty}{(q^2;q^2)_\infty} \\
&= \frac{(q;q^2)_\infty (q^8;q^8)_\infty (-q;q^2)_\infty}{(q^2;q^2)_\infty (-q;q^8)_\infty (-q^7;q^8)_\infty} \\
&= \frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty (q^8;q^8)_\infty (-q;q^8)_\infty (-q^7;q^8)_\infty} \\
&= \frac{\psi(q^4)}{f(q, q^7)},
\end{aligned}$$

as desired. □

The latter entry and the next entry are the analytic versions of the two famous Göllnitz–Gordon identities [157]. They can also be found in Slater's list [262, equations (36), (34)], but with q replaced by $-q$. The Göllnitz–Gordon identities have played a seminal role in the subsequent development of the theory of partition identities. They were first studied in this regard by H. Göllnitz [156] and by B. Gordon [158], [159]. A generalization by Andrews [10] led to a number of further discoveries culminating in [16].

Entry 1.7.12 (p. 41). *If $\psi(q)$ and $f(a,b)$ are defined by* (1.4.10) *and* (1.4.8), *respectively, then*

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} = \frac{\psi(q^4)}{f(q^3,q^5)}.$$

Proof. In (1.2.1), set $h=1$, replace q by q^2, then set $a=q^3/t$, $b=q$, and $c=0$, and lastly let $t\to 0$. Simplifying, using (1.7.14), and invoking the triple product identity (1.4.8), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} &= (q;q^2)_\infty^2 \sum_{n=0}^{\infty} \frac{q^n}{(q;q)_{2n+1}} \\ &= \frac{(q;q^2)_\infty^2}{(q;q)_\infty} f(q,q^7) \\ &= \frac{(q;q^2)_\infty(-q;q^8)_\infty(-q^7;q^8)_\infty(q^8;q^8)_\infty}{(q^2;q^2)_\infty} \\ &= \frac{(q;q^2)_\infty(-q;q^2)_\infty(q^8;q^8)_\infty}{(q^2;q^2)_\infty(-q^3;q^8)_\infty(-q^5;q^8)_\infty} \\ &= \frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty(q^8;q^8)_\infty(-q^3;q^8)_\infty(-q^5;q^8)_\infty} \\ &= \frac{\psi(q^4)}{f(q^3,q^5)},\end{aligned}$$

by (1.4.10) and (1.4.8). □

Entry 1.7.13 (p. 35). *Recall that $\varphi(-q)$ is defined in* (1.4.9). *Then*

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2}}{(q;q)_{2n+1}} = \frac{\varphi(-q^2)}{\varphi(-q)}.$$

Proof. We apply (1.2.9) with q replaced by q^2 and with a, b, and c replaced by $-q/t$, $-q^2$, and q^3, respectively. We then let t tend to 0. Thus,

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2}}{(q;q)_{2n+1}} = \frac{1}{1-q}\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2}}{(q^2;q^2)_n(q^3;q^2)_n}$$

$$
\begin{aligned}
&= \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \\
&= \frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty}\frac{(-q;q)_\infty}{(q;q)_\infty} \\
&= \frac{\varphi(-q^2)}{\varphi(-q)},
\end{aligned}
$$

by (1.4.9). □

The next entry is also a special case of Lebesgue's identity [200], [18, p. 21, Corollary 2.7]. One can also find it in Slater's paper [262, equation (12)].

Entry 1.7.14 (p. 34). *With $\varphi(q)$ as in the previous entry,*

$$
\sum_{n=0}^{\infty} \frac{(-1)_n q^{n(n+1)/2}}{(q)_n} = \frac{\varphi(-q^2)}{\varphi(-q)}.
$$

Proof. In (1.2.1), set $h = 1$, $a = -q/t$, and $c = -bt$, and then let $t \to 0$. After simplification and the use of (1.2.3), we find that [18, p. 21]

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{(b;q)_n q^{n(n+1)/2}}{(q;q)_n} &= (b;q)_\infty(-q;q)_\infty \sum_{n=0}^{\infty} \frac{b^n}{(q^2;q^2)_n} \\
&= \frac{(b;q)_\infty(-q;q)_\infty}{(b;q^2)_\infty} \\
&= (bq;q^2)_\infty(-q;q)_\infty.
\end{aligned}
$$

Setting $b = -1$ above, we find that

$$
\sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n(n+1)/2}}{(q;q)_n} = (-q;q^2)_\infty(-q;q)_\infty. \tag{1.7.19}
$$

Now, by Euler's identity,

$$
(-q;q^2)_\infty = \frac{(-q;q)_\infty}{(-q^2;q^2)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty(q^2;q^2)_\infty(-q^2;q^2)_\infty} = \frac{(q^2;q^2)_\infty}{(q;q)_\infty(-q^2;q^2)_\infty}. \tag{1.7.20}
$$

Thus, using (1.7.20) in (1.7.19), we find that

$$
\sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n(n+1)/2}}{(q;q)_n} = \frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty}\frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{\varphi(-q^2)}{\varphi(-q)},
$$

by (1.4.9). □

Entry 1.7.15 (p. 28). *If*

$$\phi(a,q) := \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n},$$

then

$$\phi(a,q)\phi(b,1/q) = \sum_{n=0}^{\infty} \frac{(b/(aq^n);1/q)_n a^n q^{n^2}}{(q;q)_n}.$$

This is a purely formal power series identity in the variables a and b. There are no values of q for which all the series in the identity converge.

Proof. Formally, after algebraic simplification,

$$\phi(a,q)\phi(b,1/q) = \sum_{j,n=0}^{\infty} \frac{(-1)^n a^j b^n q^{j^2-n(n-1)/2}}{(q;q)_j (q;q)_n}. \tag{1.7.21}$$

On the other hand, by (1.7.3),

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(b/(aq^n);1/q)_n a^n q^{n^2}}{(q;q)_n} &= \sum_{n=0}^{\infty} \frac{(-1)^n b^n q^{-n(n-1)/2}(aq^n/b;q)_n}{(q;q)_n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n b^n q^{-n(n-1)/2}}{(q;q)_n} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^j (aq^n/b)^j q^{j(j-1)/2} \\ &= \sum_{j,n=0}^{\infty} \frac{(-1)^n a^j b^n q^{j^2-n(n-1)/2}}{(q;q)_j (q;q)_n},\end{aligned} \tag{1.7.22}$$

where in the last step we replaced n by $n+j$ after inverting the order of summation. Comparing (1.7.21) and (1.7.22), we see that we have completed the proof. □

Entry 1.7.16 (p. 28). *Let a and b be any complex numbers, and suppose that $|xy| < 1$. If*

$$\phi(a,x,y) := \sum_{n=0}^{\infty} \frac{a^n x^{n(n+1)/2}}{(xy;xy)_n},$$

then

$$\phi(a,x,y)\phi(b,y,x) = \sum_{n=0}^{\infty} \frac{(ax+by^n)(ax^2+by^{n-1})\cdots(ax^n+by)}{(xy;xy)_n}.$$

Proof. Applying the q-binomial theorem (1.7.3), we find that

$$\sum_{n=0}^{\infty} \frac{(ax+by^n)(ax^2+by^{n-1})\cdots(ax^n+by)}{(xy;xy)_n}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{y^{-n(n+1)/2}(axy+by^{n+1})(ax^2y^2+by^{n+1})\cdots(ax^ny^n+by^{n+1})}{(xy;xy)_n} \\
&= \sum_{n=0}^{\infty} \frac{y^{-n(n+1)/2}}{(xy;xy)_n} \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{xy} (by^{n+1})^{n-j}(axy)^j(xy)^{j(j-1)/2} \\
&= \sum_{n,j=0}^{\infty} \frac{a^j b^n x^{j(j+1)/2} y^{n(n+1)/2}}{(xy;xy)_j (xy;xy)_n} \\
&= \phi(a,x,y)\phi(b,y,x),
\end{aligned}
$$

where in the penultimate line we inverted the order of summation and then replaced n by $n+j$. □

Entry 1.7.17 (p. 57). *If*

$$\phi(a) := \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q^2;q^2)_n},$$

then

$$\phi(a)\phi(b) = \sum_{n=0}^{\infty} \frac{(a+bq^{n-1})(a+bq^{n-3})\cdots(a+bq^{1-n})q^{n(n+1)/2}}{(q^2;q^2)_n}.$$

Proof. Set $x=y=q$ in Entry 1.7.16. □

Entry 1.7.18 (p. 31). *The expression*

$$\frac{1}{1-\alpha} + \sum_{n=1}^{\infty} \frac{\beta^n}{(1-\alpha x^n)(1-\alpha x^{n-1}y)\cdots(1-\alpha y^n)}$$

is symmetric in α *and* β.

Proof. Set $q = y/x$. Then, by (1.7.4),

$$
\begin{aligned}
&\frac{1}{1-\alpha} + \sum_{n=1}^{\infty} \frac{\beta^n}{(1-\alpha x^n)(1-\alpha x^{n-1}y)\cdots(1-\alpha y^n)} \\
&\qquad = \frac{1}{1-\alpha} + \sum_{n=1}^{\infty} \frac{\beta^n}{(1-\alpha x^n)(1-\alpha x^n q)\cdots(1-\alpha x^n q^n)} \\
&\qquad = \sum_{n=0}^{\infty} \frac{\beta^n}{(\alpha x^n;q)_{n+1}} \\
&\qquad = \sum_{n,m=0}^{\infty} \beta^n \alpha^m \begin{bmatrix} n+m \\ m \end{bmatrix} x^{nm},
\end{aligned}
$$

which is easily seen to be symmetric in α and β. □

For a combinatorial proof of Entry 1.7.18, see [19, pp. 106–107], [75, p. 182].

Entry 1.7.19 (p. 26). *Define the coefficients* $c_n, n \geq 0$, *by*

$$\sum_{n=0}^{\infty} c_n \lambda^n := \frac{(-a\lambda)_\infty}{(b\lambda)_\infty (c\lambda)_\infty}.$$

Then

$$\sum_{n=0}^{\infty} c_n q^{n(n+1)/2} = (-cq)_\infty \sum_{n=0}^{\infty} \frac{(-a/b)_n b^n q^{n(n+1)/2}}{(q)_n (-cq)_n}. \tag{1.7.23}$$

Proof. By (1.2.2) and (1.2.3),

$$\begin{aligned}
\sum_{n=0}^{\infty} c_n \lambda^n &= \frac{(-a\lambda)_\infty}{(b\lambda)_\infty} \frac{1}{(c\lambda)_\infty} \\
&= \sum_{m=0}^{\infty} \frac{(-a/b)_m (b\lambda)^m}{(q)_m} \sum_{k=0}^{\infty} \frac{(c\lambda)^k}{(q)_k} \\
&= \sum_{n=0}^{\infty} \lambda^n \sum_{m+k=n} \frac{(-a/b)_m b^m c^k}{(q)_m (q)_k}.
\end{aligned}$$

Hence, for $n \geq 0$,

$$c_n = \sum_{m+k=n} \frac{(-a/b)_m b^m c^k}{(q)_m (q)_k}.$$

Therefore, by (1.2.4),

$$\begin{aligned}
\sum_{n=0}^{\infty} c_n q^{n(n+1)/2} &= \sum_{n=0}^{\infty} \sum_{m+k=n} \frac{(-a/b)_m b^m c^k q^{(m+k)(m+k+1)/2}}{(q)_m (q)_k} \\
&= \sum_{m=0}^{\infty} \frac{(-a/b)_m b^m q^{m(m+1)/2}}{(q)_m} \sum_{k=0}^{\infty} \frac{(cq^{m+1})^k q^{k(k-1)/2}}{(q)_k} \\
&= \sum_{m=0}^{\infty} \frac{(-a/b)_m b^m q^{m(m+1)/2}}{(q)_m} (-cq^{m+1})_\infty \\
&= (-cq)_\infty \sum_{m=0}^{\infty} \frac{(-a/b)_m b^m q^{m(m+1)/2}}{(q)_m (-cq)_m},
\end{aligned}$$

which is what we wanted to prove. ∎

If we set $a = 0$ in (1.7.23), we see that Entry 6.2.4 in our first volume on the lost notebook [31, p. 148] is an immediate corollary of Entry 1.7.19. Furthermore, in the notation of Entry 6.2.4 of [31], the claim that $(-bq)_\infty G(a, b)$ is symmetric in a and b, as asserted in [244, p. 42], is also immediate from Entry 1.7.19.

Entry 1.7.20 (p. 57). *Let m and n be nonnegative integers. Let*

$$\phi(a) := \sum_{j=0}^{\infty} \frac{a^j q^{mj^2+nj}}{(q)_j} \qquad \text{and} \qquad \psi(a) := \frac{\phi(aq)}{\phi(a)}.$$

Then

$$\phi(a) = \phi(aq) + aq^{m+n}\phi(aq^{2m}) \tag{1.7.24}$$

and

$$1 = \psi(a) + aq^{m+n}\psi(a)\psi(aq)\psi(aq^2)\cdots\psi(aq^{2m-1}).$$

Proof. We have

$$\begin{aligned}
\phi(a) - \phi(aq) &= \sum_{j=0}^{\infty} \frac{a^j q^{mj^2+nj}(1-q^j)}{(q)_j} \\
&= \sum_{j=1}^{\infty} \frac{a^j q^{mj^2+nj}}{(q)_{j-1}} \\
&= aq^{m+n} \sum_{j=0}^{\infty} \frac{a^j q^{mj^2+2mj+nj}}{(q)_j} \\
&= aq^{m+n}\phi(aq^{2m}),
\end{aligned}$$

which is the first assertion.

Upon dividing both sides of (1.7.24) by $\phi(a)$ and noting that

$$\frac{\phi(aq^{2m})}{\phi(a)} = \psi(a)\psi(aq)\psi(aq^2)\cdots\psi(a^{2m-1}),$$

we obtain the second assertion. □

Entry 1.7.21 (p. 10). *For $|xy| < 1$ and a and b arbitrary,*

$$\sum_{n=0}^{\infty} \frac{(ab)^n (xy)^{n(n+1)/2} y^{-n}}{(ax;x)_n (by;y)_n} = 1 - b + b\sum_{n=0}^{\infty} \frac{(ab)^n (xy)^{n(n+1)/2}}{(ax;x)_{n+1}(by;y)_n}.$$

Proof. We have

$$\begin{aligned}
&1 - b + b\sum_{n=0}^{\infty} \frac{(ab)^n (xy)^{n(n+1)/2}}{(ax;x)_{n+1}(by;y)_n} \\
&\qquad = 1 - b + b\sum_{n=0}^{\infty} \frac{(ab)^n (xy)^{n(n+1)/2}}{(ax;x)_n (by;y)_n}\left(1 + \frac{ax^{n+1}}{1-ax^{n+1}}\right) \\
&\qquad = 1 - b + b\sum_{n=0}^{\infty} \frac{(ab)^n (xy)^{n(n+1)/2}}{(ax;x)_n (by;y)_n} + \sum_{n=0}^{\infty} \frac{(ab)^{n+1} (xy)^{(n+1)(n+2)/2} y^{-n-1}}{(ax;x)_{n+1}(by;y)_n}
\end{aligned}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(ab)^n b(xy)^{n(n+1)/2}}{(ax;x)_n(by;y)_n} + \sum_{n=1}^{\infty} \frac{(ab)^n (xy)^{n(n+1)/2} y^{-n}}{(ax;x)_n(by;y)_{n-1}}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(ab)^n (xy)^{n(n+1)/2} y^{-n}}{(ax;x)_n(by;y)_n} \left(by^n + (1 - by^n)\right)$$

$$= \sum_{n=0}^{\infty} \frac{(ab)^n (xy)^{n(n+1)/2} y^{-n}}{(ax;x)_n(by;y)_n}.$$

Thus, the proof is complete. □

2

The Sears–Thomae Transformation

2.1 Introduction

In this chapter, we consider those identities in the lost notebook most closely tied to the transformation

$$ {}_3\phi_2\left(\begin{matrix} a, b, c \\ d, e \end{matrix}; q, \frac{de}{abc}\right) = \frac{(e/a)_\infty (de/(bc))_\infty}{(e)_\infty (de/(abc))_\infty} \, {}_3\phi_2\left(\begin{matrix} a, d/b, d/c \\ d, de/(bc) \end{matrix}; q, \frac{e}{a}\right), \tag{2.1.1} $$

where

$$ {}_{r+1}\phi_r\left(\begin{matrix} a_0, a_1, \dots, a_r \\ b_1, \dots, b_r \end{matrix}; q, t\right) = \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \cdots (a_r)_n}{(q)_n (b_1)_n \cdots (b_r)_n} t^n. \tag{2.1.2} $$

Identity (2.1.1) was first proved by D.B. Sears [256] as a q-analogue of a result of J. Thomae [270].

It must be pointed out that in subsequent chapters use is often made of (2.1.1). In this chapter, we examine only those identities that are primarily a consequence of (2.1.1) and possibly some further elementary transformations.

Before we proceed to Ramanujan's discoveries, we note a limiting case [15, Lemma] of (2.1.1) when $c \to \infty$, namely,

$$ \sum_{n=0}^{\infty} \frac{(-1)^n (a)_n (b)_n}{(q)_n (d)_n (e)_n} \left(\frac{de}{ab}\right)^n q^{n(n-1)/2} = \frac{(e/a)_\infty}{(e)_\infty} \sum_{n=0}^{\infty} \frac{(a)_n (d/b)_n}{(q)_n (d)_n} \left(\frac{e}{a}\right)^n. \tag{2.1.3} $$

2.2 Direct Corollaries of (2.1.1) and (2.1.3)

Entry 2.2.1 (p. 1). *For* $0 < |aq| < 1$,

$$ \sum_{n=0}^{\infty} \frac{(-q;q)_{2n} q^n}{(aq;q^2)_{n+1} (q/a;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-q/a;q)_{2n} a^n q^n}{(q;q^2)_{n+1} (q/a;q^2)_{n+1}}. $$

G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook: Part II*,
DOI 10.1007/978-0-387-77766-5_3,

Proof. In (2.1.1), replace q by q^2 and then replace a, b, c, d, and e by q^2, $-q$, $-q^2$, q^3/a, and aq^3, respectively. Then multiply both sides by $1/(1-aq)$ and by $1/(1-q/a)$. □

Entry 2.2.2 (p. 8). *We have*

$$\sum_{n=0}^{\infty}\frac{(-q^2;q^2)_n q^{n+1}}{(q;q^2)_{n+1}}=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{(n+1)^2}}{(q;q^2)_{n+1}^2}.$$

Proof. In (2.1.3), replace q by q^2, and then replace a, b, d, and e by q^2, $-q$, q^3, and q^3, respectively. Finally, multiply both sides by $q/(1-q)^2$. □

Entry 2.2.3 (p. 42). *For arbitrary complex numbers a, b, and c,*

$$\frac{(aq)_\infty}{(-bq)_\infty}\sum_{n=0}^{\infty}\frac{(-aq/b)_n b^n q^{n(n+1)/2}}{(q)_n(-cq)_n}=(aq)_\infty\sum_{n=0}^{\infty}\frac{(bc/a)_n a^n q^{n^2+n}}{(q)_n(-bq)_n(-cq)_n}.$$

We have recorded Entry 2.2.3 as Ramanujan recorded it, that is, with a common factor $(aq)_\infty$ on each side.

Proof. In (2.1.3), let $a\to\infty$, and then replace b, d, and e by bc/a, $-cq$, and $-bq$, respectively. □

Entry 2.2.4 (p. 42). *Let a and b be arbitrary complex numbers. Then*

$$\sum_{n=0}^{\infty}\frac{a^n q^{n(n+1)/2}}{(q)_n(-bq)_n}=(-aq)_\infty\sum_{n=0}^{\infty}\frac{(-1)^n(ab)^n q^{n(3n+1)/2}}{(q)_n(-aq)_n(-bq)_n}.$$

Proof. In Entry 2.2.3, let $a\to 0$, and then replace b by a and c by b. □

2.3 Extended Corollaries of (2.1.1) and (2.1.3)

Entry 2.3.1 (p. 37). *Let a be arbitrary and suppose that $|bq|<1$. Then*

$$\sum_{n=0}^{\infty}\frac{(ab)^n q^{n^2}}{(aq)_n(bq)_n}=1+a\sum_{n=1}^{\infty}\frac{b^n q^n}{(aq)_n},\tag{2.3.1}$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n} q^{n^2}}{(a^2q^2;q^2)_n}=1-a\sum_{n=1}^{\infty}\frac{a^n q^n}{(-aq)_n}.\tag{2.3.2}$$

Proof. In (2.1.1), replace a, b, c, d, and e by q, $1/\tau$, $1/\tau$, aq, and bq, respectively. Then let $\tau\to 0$. After simplification, this yields

$$\sum_{n=0}^{\infty}\frac{(ab)^n q^{n^2}}{(aq)_n(bq)_n}=(1-b)\sum_{n=0}^{\infty}\frac{b^n}{(aq)_n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{b^n}{(aq)_n} - \sum_{n=1}^{\infty} \frac{b^n}{(aq)_{n-1}}$$
$$= 1 + a \sum_{n=1}^{\infty} \frac{b^n q^n}{(aq)_n},$$

and so (2.3.1) has been proved.

The identity (2.3.2) is obtained by setting $a = -b$ in (2.3.1) and then replacing b by a. $\square$

Entry 2.3.2 (p. 34). *Recall that $\varphi(q)$ is defined by* (1.4.9). *Then*

$$\sum_{n=0}^{\infty} \frac{(-1;q)_n^2 q^{n(n+1)/2}}{(q;q)_n (q;q^2)_n} = \frac{\varphi(q)}{\varphi(-q)}.$$

Proof. In (2.1.3), replace a, b, d, and e by -1, -1, $q^{1/2}$, and $-q^{1/2}$, respectively. Consequently, after simplification and the use of the q-binomial theorem (1.2.2),

$$\begin{aligned}
S &:= \sum_{n=0}^{\infty} \frac{(-1;q)_n^2 q^{n(n+1)/2}}{(q;q)_n (q;q^2)_n} \\
&= \frac{(q^{1/2};q)_\infty}{(-q^{1/2};q)_\infty} \sum_{n=0}^{\infty} \frac{(-1;q)_n (-q^{1/2};q)_n q^{n/2}}{(q;q)_n (q^{1/2};q)_n} \\
&= \frac{(q^{1/2};q)_\infty}{(-q^{1/2};q)_\infty} \sum_{n=0}^{\infty} \frac{(-1;q^{1/2})_{2n} q^{n/2}}{(q^{1/2};q^{1/2})_{2n}} \\
&= \frac{(q^{1/2};q)_\infty}{2(-q^{1/2};q)_\infty} \sum_{n=0}^{\infty} (1+(-1)^n) \frac{(-1;q^{1/2})_n q^{n/4}}{(q^{1/2};q^{1/2})_n} \\
&= \frac{(q^{1/2};q)_\infty}{2(-q^{1/2};q)_\infty} \left(\frac{(-q^{1/4};q^{1/2})_\infty}{(q^{1/4};q^{1/2})_\infty} + \frac{(q^{1/4};q^{1/2})_\infty}{(-q^{1/4};q^{1/2})_\infty} \right).
\end{aligned}$$

Now put the fractions on the right side above under a common denominator, multiply both the numerator and denominator by $(q^{1/2};q^{1/2})_\infty$, and simplify to arrive at

$$\begin{aligned}
S &= \frac{1}{2(q;q^2)_\infty (q;q)_\infty} \Big((-q^{1/4};q^{1/2})_\infty^2 (q^{1/2};q^{1/2})_\infty \\
&\qquad + (q^{1/4};q^{1/2})_\infty^2 (q^{1/2};q^{1/2})_\infty \Big) \\
&= \frac{1}{2(q;q^2)_\infty (q;q)_\infty} \left(\sum_{n=-\infty}^{\infty} q^{n^2/4} + \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/4} \right) \qquad (2.3.3) \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} q^{n^2}
\end{aligned}$$

$$= \frac{\varphi(q)}{\varphi(-q)},$$

where in the antepenultimate line we used the Jacobi triple product identity (1.4.8), and then afterward used Euler's identity and the product representation from (1.4.9). This completes the proof. □

Entry 2.3.3 (p. 35). *If* $\varphi(q)$ *and* $\psi(q)$ *are defined in* (1.4.9) *and* (1.4.10), *respectively, then*

$$\sum_{n=0}^{\infty} \frac{(-q;q)_n^2 q^{n(n+1)/2}}{(q;q)_n (q;q^2)_{n+1}} = \frac{\psi(q^2)}{\varphi(-q)}.$$

Proof. In (2.1.3), set $a = b = -q$, $d = q^{3/2}$, and $e = -q^{3/2}$, and multiply both sides of the resulting identity by $1/(1-q)$. We proceed in the same fashion as in the previous proof. In particular, we use the calculation (2.3.3). Consequently,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-q;q)_n^2 q^{n(n+1)/2}}{(q;q)_n (q;q^2)_{n+1}} &= \frac{(q^{1/2};q)_\infty}{(-q^{1/2};q)_\infty} \sum_{n=0}^{\infty} \frac{(-q;q)_n (-q^{1/2};q)_n q^{n/2}}{(q;q)_n (q^{1/2};q)_{n+1}} \\
&= \frac{(q^{1/2};q)_\infty}{2(-q^{1/2};q)_\infty} \sum_{n=0}^{\infty} \frac{(-1;q^{1/2})_{2n+1} q^{n/2}}{(q^{1/2};q^{1/2})_{2n+1}} \\
&= \frac{(q^{1/2};q)_\infty q^{-1/4}}{4(-q^{1/2};q)_\infty} \sum_{n=0}^{\infty} (1-(-1)^n) \frac{(-1;q^{1/2})_n q^{n/4}}{(q^{1/2};q^{1/2})_n} \\
&= \frac{q^{-1/4}}{4(q;q^2)_\infty (q;q)_\infty} \left(\sum_{n=-\infty}^{\infty} q^{n^2/4} - \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2/4} \right) \\
&= \frac{(-q;q)_\infty q^{-1/4}}{2(q;q)_\infty} \sum_{n=-\infty}^{\infty} q^{(2n+1)^2/4} \\
&= \frac{(-q;q)_\infty}{(q;q)_\infty} \sum_{n=0}^{\infty} q^{n^2+n} \\
&= \frac{\psi(q^2)}{\varphi(-q)},
\end{aligned}$$

where we employed the product representation of $\varphi(-q)$ given in (1.4.9). □

Entry 2.3.4 (p. 35). *Recall that* $f(a,b)$ *is defined in* (1.4.8). *Then*

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^4)_n q^{n^2+n}}{(q^2;q^2)_n (q^2;q^4)_n (1-q^{2n+1})} = \frac{f(-q^6,-q^{10}) + qf(-q^2,-q^{14})}{\varphi(-q^2)}. \tag{2.3.4}$$

Proof. We prove Entry 2.3.4 with q replaced by $-q$. Applying (2.1.3) below with q replaced by q^2, and then setting $a = iq$, $b = -iq$, $d = q$, and $e = -q^3$, we find that

$$\begin{aligned}
S &:= \sum_{n=0}^{\infty} \frac{(-q^2;q^4)_n q^{n^2+n}}{(q^2;q^2)_n (q^2;q^4)_n (1+q^{2n+1})} \\
&= \frac{1}{1+q} \sum_{n=0}^{\infty} \frac{(iq;q^2)_n (-iq;q^2)_n q^{n^2+n}}{(q^2;q^2)_n (-q^3;q^2)_n (q;q^2)_n} \\
&= \frac{1}{1+q} \frac{(iq^2;q^2)_\infty}{(-q^3;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(iq;q^2)_n (i;q^2)_n (iq^2)^n}{(q^2;q^2)_n (q;q^2)_n} \\
&= \frac{(iq^2;q^2)_\infty}{(-q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(i;q)_{2n} (iq^2)^n}{(q;q)_{2n}} \\
&= \frac{(iq^2;q^2)_\infty}{2(-q;q^2)_\infty} \sum_{n=0}^{\infty} (1+(-1)^n) \frac{(i;q)_n i^{n/2} q^n}{(q;q)_n} \\
&= \frac{(iq^2;q^2)_\infty}{2(-q;q^2)_\infty} \left(\frac{(i^{3/2}q;q)_\infty}{(i^{1/2}q;q)_\infty} + \frac{(-i^{3/2}q;q)_\infty}{(-i^{1/2}q;q)_\infty} \right),
\end{aligned}$$

where we have applied the q-binomial theorem (1.2.2). Putting the terms on the far right side above under a common denominator and applying the Jacobi triple product identity (1.4.8), we find that

$$\begin{aligned}
S &= \frac{(iq^2;q^2)_\infty}{2(-q;q^2)_\infty (iq^2;q^2)_\infty} \Big((-i^{1/2}q;q)_\infty (-i^{-1/2}q;q)_\infty \\
&\qquad + (i^{1/2}q;q)_\infty (i^{-1/2}q;q)_\infty \Big) \\
&= \frac{1}{2(q;q)_\infty (-q;q^2)_\infty} \left(\frac{(-i^{1/2};q)_\infty (-i^{-1/2}q;q)_\infty (q;q)_\infty}{1+i^{1/2}} \right. \\
&\qquad \left. + \frac{(i^{1/2};q)_\infty (i^{-1/2}q;q)_\infty (q;q)_\infty}{1-i^{1/2}} \right) \\
&= \frac{1}{2(q;q)_\infty (-q;q^2)_\infty} \left(\frac{1}{1+i^{1/2}} \sum_{n=-\infty}^{\infty} i^{n/2} q^{n(n-1)/2} \right. \\
&\qquad \left. + \frac{1}{1-i^{1/2}} \sum_{n=-\infty}^{\infty} (-1)^n i^{n/2} q^{n(n-1)/2} \right) \\
&= \frac{1}{2(1-i)(q;q)_\infty (-q;q^2)_\infty} \left((1-i^{1/2}) \sum_{n=-\infty}^{\infty} i^{n/2} q^{n(n-1)/2} \right. \\
&\qquad \left. + (1+i^{1/2}) \sum_{n=-\infty}^{\infty} (-1)^n i^{n/2} q^{n(n-1)/2} \right) \\
&= \frac{1}{(1-i)(q;q)_\infty (-q;q^2)_\infty} \left(\sum_{n=-\infty}^{\infty} i^n q^{n(2n-1)} - \sum_{n=-\infty}^{\infty} i^{n+1} q^{n(2n+1)} \right)
\end{aligned}$$

$$= \frac{1}{(1-i)(q;q)_\infty(-q;q^2)_\infty}\left(\sum_{n=-\infty}^{\infty} i^n q^{n(2n-1)} - i\sum_{n=-\infty}^{\infty}(-i)^n q^{n(2n-1)}\right),$$

where we replaced n by $-n$ in the latter sum. Dissecting each sum above into its even- and odd-indexed terms and applying the triple product identity (1.4.8) and the product representation for $\varphi(-q^2)$ given in (1.4.9), we conclude that

$$\begin{aligned}
S &= \frac{1+i}{2(q;q)_\infty(-q;q^2)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{8n^2-2n} + i\sum_{n=-\infty}^{\infty}(-1)^n q^{(2n+1)(4n+1)}\right.\\
&\qquad\left. -i\sum_{n=-\infty}^{\infty}(-1)^n q^{8n^2-2n} - \sum_{n=-\infty}^{\infty}(-1)^n q^{(2n+1)(4n+1)}\right)\\
&= \frac{1}{(q^2;q^2)_\infty(q^2;q^4)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{8n^2-2n} - \sum_{n=-\infty}^{\infty}(-1)^n q^{8n^2+6n+1}\right)\\
&= \frac{f(-q^6,-q^{10}) - qf(-q^2,-q^{14})}{\varphi(-q^2)},
\end{aligned}$$

which is (2.3.4) with q replaced by $-q$, as we had intended to prove. □

Entry 2.3.5 (p. 5). *For any complex number b and $|aq|<1$,*

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n(b^2q/a;q^2)_n a^n q^n}{(-bq;q)_{2n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n(-aq/b;q)_n b^n q^{n(n+1)/2}}{(aq;q^2)_{n+1}}.$$

Proof. After rearrangement of the series on the left-hand side above, we apply (1.2.1) with $h=2$ and a, b, c, and t replaced by b^2q/a, q, $-bq^2$, and aq, respectively, to deduce that

$$\begin{aligned}
S &:= \sum_{n=0}^{\infty}\frac{(q;q^2)_n(b^2q/a;q^2)_n a^n q^n}{(-bq;q)_{2n+1}}\\
&= \frac{1}{1+bq}\sum_{n=0}^{\infty}\frac{(b^2q/a;q^2)_n(q;q)_{2n}a^nq^n}{(q^2;q^2)_n(-bq^2;q)_{2n}}\\
&= \frac{(q;q)_\infty(b^2q^2;q^2)_\infty}{(-bq;q)_\infty(aq;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-bq;q)_n(aq;q^2)_nq^n}{(q;q)_n(b^2q^2;q^2)_n}\\
&= \frac{(q;q)_\infty(bq;q)_\infty}{(aq;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(aq;q^2)_nq^n}{(q;q)_n(bq;q)_n}\\
&= \frac{(q;q)_\infty(bq;q)_\infty}{(aq;q^2)_\infty}\frac{(\sqrt{aq};q)_\infty(-q\sqrt{aq};q)_\infty}{(q;q)_\infty(bq;q)_\infty}\sum_{n=0}^{\infty}\frac{(b\sqrt{q/a};q)_n(aq)^{n/2}}{(-q\sqrt{aq};q)_n},
\end{aligned}$$

by another application of (1.2.1), this time with $h=1$ and a, b, c, and t replaced by $-\sqrt{aq}$, $\sqrt{aq}$, bq, and q, respectively. Simplify the far right side above and then apply (2.1.3) with $a = q$, then with $d = -a^{1/2}q^{3/2}$ and $e = a^{1/2}q^{3/2}$, with b replaced by $-aq/b$, and with both sides multiplied by $1/(1-\sqrt{aq})$ and $1/(1+\sqrt{aq})$. Accordingly, we find that

$$S = \sum_{n=0}^{\infty} \frac{(b\sqrt{q/a};q)_n (aq)^{n/2}}{(-\sqrt{aq};q)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n(-aq/b;q)_n b^n q^{n(n+1)/2}}{(aq;q^2)_{n+1}},$$

which is what we wanted to prove. $\square$

3

Bilateral Series

3.1 Introduction

In this chapter, we collect a number of formulas from the lost notebook that relate to classical bilateral q-hypergeometric series, defined by [151, p. 125]

$$ {}_r\psi_r\left(\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_r \end{matrix}; q, t\right) = \sum_{n=-\infty}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(b_1, b_2, \ldots, b_r; q)_n} t^n, $$

where, for any integer n,

$$ (a_1, a_2, \ldots, a_r; q)_n = (a_1; q)_n (a_2; q)_n \cdots (a_r; q)_n, $$

with

$$ (a; q)_n = \prod_{j=0}^{\infty} \frac{1 - aq^j}{1 - aq^{j+n}}. $$

In particular,

$$ (a; q)_{-n} = \frac{(-1)^n a^{-n} q^{n(n+1)/2}}{(q/a; q)_n}. \tag{3.1.1} $$

Moreover, the quintuple product identity

$$ \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} z^{3n} (1 + zq^n) = (-z, -q/z, q; q)_\infty (qz^2, q/z^2; q^2)_\infty \tag{3.1.2} $$

fits reasonably well in this chapter. Ramanujan stated the quintuple product identity only once in his extant writings, namely, on page 207 in his lost notebook. In order to state this identity in the form given by Ramanujan, recall from (1.4.8) the definition of Ramanujan's theta function $f(a, b)$. Also, in Ramanujan's notation, set

$$ f(-q) = (q; q)_\infty. \tag{3.1.3} $$

G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook: Part II*,
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Entry 3.1.1 (p. 207). *Provided all arguments below are of modulus less than* 1, *we have*

$$\frac{f(-\lambda q,-q^2)}{f(-q,-\lambda q^2)}f(-\lambda q^3)=f(-\lambda^2q^3,-\lambda q^6)+qf(-\lambda,-\lambda^2q^9).$$

An excellent survey describing all known proofs of the quintuple product identity has been prepared by S. Cooper [140]. A finite form of the quintuple product identity was established by P. Paule [227] and later by W.Y.C. Chen, W. Chu, and N.S.S. Gu [122], and by Chu [128]. A proof of the quintuple product identity by S. Bhargava, C. Adiga, and M.S. Mahadeva Naika [82] was written after the appearance of Cooper's survey. Lastly, we remark that a bijective proof of the quintuple product identity had long been sought; S. Kim [191] has recently devised such a proof.

In Section 3.2, we examine those identities associated with Ramanujan's famous ${}_1\psi_1$ summation

$${}_1\psi_1\left(\begin{matrix}a\\b\end{matrix};q,z\right)=\frac{(q,b/a,az,q/(az);q)_\infty}{(b,q/a,z,b/(az);q)_\infty}, \tag{3.1.4}$$

where $|b/a|<|z|<1$.

Section 3.3 is devoted to a variety of formulas connected with the ${}_2\psi_2$ summation and identities proved initially by W.N. Bailey [43]. Next, we focus attention on results intimately connected with the quintuple product identity. We conclude in Section 3.6 with several identities that are clearly appropriate for this chapter but do not fit into the other sections.

3.2 Background

The work in this chapter is primarily based on the celebrated ${}_1\psi_1$ summation (3.1.4) and the quintuple product identity (3.1.2) as well as three lesser-known results of W.N. Bailey [42]. Since the ${}_1\psi_1$ summation is central to this chapter and is one of Ramanujan's most famous theorems, it seems appropriate here to give a brief history of this epic theorem.

The ${}_1\psi_1$ summation theorem was first recorded by Ramanujan in his second notebook [243, Chapter 16, Entry 17], [54, p. 32]. However, because his notebooks were not published until 1957, it was not brought before the mathematical public until 1940, when G.H. Hardy recorded Ramanujan's ${}_1\psi_1$ summation theorem in his treatise on Ramanujan's work [174, pp. 222–223]. Subsequently, the first published proofs were given in 1949 and 1950 by W. Hahn [172] and M. Jackson [185], respectively. Since these first two proofs, several others have been published, namely, by Andrews [11], [12], M.E.H. Ismail [184], Andrews and R. Askey [29], Askey [39], C. Adiga, Berndt, Bhargava, and G.N. Watson [3], K.W.J. Kadell [187], N.J. Fine [149, equation(18.3)],

K. Mimachi [221], M. Schlosser [253], S.H. Chan [116], S. Corteel and J. Lovejoy [144], Corteel [143], A.J. Yee [285], W. Chu [127], and V.J.W. Guo and Schlosser [169]. The proof given in [3] and reproduced in [54, Entry 17, p. 32] was, in fact, first given in lectures at the University of Mysore by K. Venkatachaliengar in the 1960s and appeared later in his monograph [272, p. 30]. The proof of Chan [116] employs partial fractions, which appear to have been central in much of Ramanujan's work. The most succinct proof of (3.1.4) was given by Ismail [184]. It consists in the observation that if $b = q^m$ for any integer $m > 0$, then (3.1.4) is merely an instance of (1.2.2). However, this completely proves (3.1.4), because the identity holds on a convergent sequence within the domain of analyticity $|b| < 1$. W.N. Bailey's [43] proof of (3.1.2) is quite simple. Let us denote the right-hand side by $r(z)$. Bailey observes that $r(z)$ satisfies the q-difference equation

$$r(z) = z^3 q\, r(zq). \tag{3.2.1}$$

He then expands $r(z)$ in a Laurent series that yields the left-hand side up to a constant term $c_0(q)$. By setting $z = 1$ and invoking the Jacobi triple product identity (1.4.8), he concludes that $c_0(q) = 1$. The proofs by Corteel [143] and Yee [285] are combinatorial. In particular, Yee devised a bijection between the partitions generated on each side of the ${}_1\psi_1$ identity. Chu's proof [127] using the classical tool of partial summation is also markedly different from other proofs. There is also an unpublished proof by Z.-G. Liu [212]. W.P. Johnson [186], in a delightful historical article, points out that had not Cauchy made a mistake, he could have discovered the ${}_1\psi_1$ summation theorem in 1843. Lastly, Schlosser [254] has derived a noncommutative version of Ramanujan's ${}_1\psi_1$ summation formula.

The aforementioned three formulas of Bailey [42] are given by

$$\begin{aligned} &{}_2\psi_2\left(\begin{matrix}\alpha, \beta, \\ \gamma, \delta\end{matrix}; q, z\right) \\ &\quad = \frac{(\delta q/(\alpha\beta z), \gamma/\beta, \alpha z, \delta/\alpha; q)_\infty}{(q/\beta, \gamma\delta/(\alpha\beta z), \delta, z; q)_\infty}\, {}_2\psi_2\left(\begin{matrix}\alpha\beta z/\delta, \alpha \\ \alpha z, \quad \gamma\end{matrix}; q, \frac{\delta}{\alpha}\right), \end{aligned} \tag{3.2.2}$$

$$\begin{aligned} &{}_2\psi_2\left(\begin{matrix}\alpha, \beta, \\ \gamma, \delta\end{matrix}; q, z\right) \\ &\quad = \frac{(\alpha z, \beta z, \gamma q/(\alpha\beta z), \delta q/(\alpha\beta z); q)_\infty}{(q/\alpha, q/\beta, \gamma, \delta; q)_\infty}\, {}_2\psi_2\left(\begin{matrix}\alpha\beta z/\gamma, \alpha\beta z/\delta \\ \alpha z, \quad \beta z\end{matrix}; q, \frac{\gamma\delta}{\alpha\beta z}\right), \end{aligned} \tag{3.2.3}$$

and

$$\begin{aligned} {}_2\psi_2\left(\begin{matrix}e, \quad f \\ aq/c, aq/d\end{matrix}; q, \frac{aq}{ef}\right) &= \frac{(q/c, q/d, aq/e, aq/f; q)_\infty}{(aq, q/a, aq/(cd), aq/(ef); q)_\infty} \\ &\times \sum_{n=-\infty}^{\infty} \frac{(q\sqrt{a}, -q\sqrt{a}, c, d, e, f; q)_n}{(\sqrt{a}, -\sqrt{a}, aq/c, aq/d, aq/e, aq/f; q)_n} \left(\frac{a^3 q}{cdef}\right)^n q^{n^2}. \end{aligned} \tag{3.2.4}$$

Bailey [42] deduced (3.2.2) by multiplying together two instances of an identity equivalent to (3.1.4). However, one can also give a proof in the spirit of Ismail's aforementioned proof [184]. Namely, if $\gamma = q^m$, with m being a positive integer, then (3.2.2) reduces to an instance of (1.2.9). This proves (3.2.2) in full, because the identity holds on a convergent sequence within the domain of analyticity $|\gamma| < 1$.

Bailey [42] observed that (3.2.3) follows by applying (3.2.2) to itself.

Finally, (3.2.4) follows from equation (12.2.1) of Part I [31, p. 262]. To see this, replace c by aq/γ. Now, if $\gamma = q^m$, where m is a positive integer, then (3.2.4) reduces to an instance of (12.2.1) from Part I. This proves (3.2.4) in general, since the identity holds on a convergent sequence within the domain of analyticity $|\gamma| < 1$.

3.3 The ${}_1\psi_1$ Identity

We begin by noting that in fact Ramanujan had a disguised form of (3.1.4) in full generality in the lost notebook.

Entry 3.3.1 (p. 7). *For* $|abq^2/c| < |bq| < 1$,

$$\sum_{n=0}^{\infty} \frac{(cq/b;q^2)_n}{(aq;q^2)_{n+1}} b^n q^n - \sum_{n=0}^{\infty} \frac{(q/a;q^2)_n}{(bq/c;q^2)_{n+1}} \frac{a^n q^n}{c^{n+1}}$$
$$= (1-c^{-1}) \frac{(q^2/c, abq^2/c, q^2, cq^2;q^2)_\infty}{(aq/c, bq/c, aq, bq;q^2)_\infty}. \tag{3.3.1}$$

Proof. In (3.1.4), replace q by q^2, and then set $z = bq$, $a = cq/b$, and $b = aq^3$; then split the ${}_1\psi_1$ series into two sums, the first with all the nonnegative indices, and the second with the negative indices. Finally, multiply both sides by $1/(1-aq)$. Applying (3.1.1) to the negative indices, we find that the left-hand side of (3.1.4) becomes

$$\frac{1}{1-aq} \sum_{n=-\infty}^{\infty} \frac{(cq/b;q^2)_n}{(aq^3;q^2)_n} b^n q^n$$
$$= \sum_{n=0}^{\infty} \frac{(cq/b;q^2)_n}{(aq;q^2)_{n+1}} b^n q^n + \frac{1}{1-aq} \sum_{n=1}^{\infty} \frac{(cq/b)^{-n}(q^2/(aq^3);q^2)_n}{(aq^3)^{-n}(bq^2/(cq);q^2)_n} b^{-n} q^{-n}$$
$$= \sum_{n=0}^{\infty} \frac{(cq/b;q^2)_n}{(aq;q^2)_{n+1}} b^n q^n - \sum_{n=1}^{\infty} \frac{(q/a;q^2)_{n-1}}{(bq/c;q^2)_n} \frac{(aq)^{n-1}}{c^n}. \tag{3.3.2}$$

This reduces to the left-hand side of (3.3.1). It is easily checked that the right-hand side of (3.1.4) reduces to the right-hand side of the above. Thus, this entry is, indeed, identity (3.1.4) in full generality. ∎

Entry 3.3.2 (p. 370). *If $k = ab$, $|k| < 1$, and n is any complex number, then*

$$\frac{(-a/n;k)_\infty(-bn;k)_\infty(k;k)_\infty^2}{(a;k)_\infty(b;k)_\infty(-nk;k)_\infty(-k/n;k)_\infty} = 1 + (n+1)\sum_{j=1}^{\infty}\left(\frac{a^j}{n+k^j} + \frac{b^j}{1+nk^j}\right). \tag{3.3.3}$$

Proof. On page 354 of his second notebook, Ramanujan claimed that

$$\frac{f(a/n,bn)f^3(-ab)}{n\,f(-a,-b)f(nab,1/n)} = \frac{1}{n+1} + \sum_{j=1}^{\infty}\left(\frac{a^j}{n+(ab)^j} + \frac{b^j}{1+n(ab)^j}\right), \tag{3.3.4}$$

where $f(a,b)$ is defined by (1.4.8) and $f(-q)$ is defined by (3.1.3). The identity (3.3.4) was proved by Berndt in [56, p. 152, Entry 17]. Comparing (3.3.3) with (3.3.4), we see that we need to show that

$$\frac{(-a/n;k)_\infty(-bn;k)_\infty(k;k)_\infty^2}{(a;k)_\infty(b;k)_\infty(-nk;k)_\infty(-k/n;k)_\infty} = \frac{(n+1)f(a/n,bn)f^3(-k)}{nf(-a,-b)f(nk,1/n)}. \tag{3.3.5}$$

However, with the use of the Jacobi triple product identity (1.4.8), this is an easy exercise. □

The next result is the same as Entry 3.3.2, but in a different notation. In this entry on page 312, Ramanujan uses the notation $f(x,y)$, but it does not denote Ramanujan's usual theta function defined in (1.4.8). Therefore, we have replaced $f(x,y)$ by $F(x,y)$ below.

Entry 3.3.3 (p. 312). *For any complex numbers x and y with $|x| < 1$, let*

$$F(x,y) := \sum_{k=-\infty}^{\infty} x^{k^2}y^k.$$

Then, for complex numbers a, b, and n with $|ab| < 1$,

$$\frac{F(\sqrt{ab},n\sqrt{b/a})(ab;ab)_\infty^3}{n\,F(\sqrt{ab},-\sqrt{b/a})F(\sqrt{ab},n\sqrt{ab})} = \frac{1}{n+1} + \sum_{j=1}^{\infty}\left(\frac{a^j}{n+(ab)^j} + \frac{b^j}{1+n(ab)^j}\right). \tag{3.3.6}$$

Proof. First observe that $F(x,y) = f(xy,x/y)$, in the notation (1.4.8). Thus, if we rewrite (3.3.6) in the notation of $f(x,y)$, we obtain precisely (3.3.4). Hence, (3.3.6) is equivalent to Entry 3.3.2, and so the proof is complete. □

Entry 3.3.4 (p. 47).

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} = \sum_{n=0}^{\infty}\frac{q^n}{1-q^{3n+1}} - \sum_{n=0}^{\infty}\frac{q^{2n+1}}{1-q^{3n+2}} \tag{3.3.7}$$

$$= \sum_{n=0}^{\infty}\frac{1+q^{3n+1}}{1-q^{3n+1}}q^{3n^2+2n} - \sum_{n=0}^{\infty}\frac{1+q^{3n+2}}{1-q^{3n+2}}q^{3n^2+4n+1}. \tag{3.3.8}$$

Proof. We first show that the right-hand side in the first line of (3.3.7) reduces to an instance of (3.1.4). More precisely,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^n}{1-q^{3n+1}}-\sum_{n=1}^{\infty}\frac{q^{2n-1}}{1-q^{3n-1}}&=\sum_{n=-\infty}^{\infty}\frac{q^n}{1-q^{3n+1}}\\&=\frac{1}{1-q}{}_1\psi_1\left(\begin{matrix}q\\q^4\end{matrix};q^3,q\right)\\&=\frac{(q^3;q^3)_\infty^2(q;q^3)_\infty(q^2;q^3)_\infty}{(q;q^3)_\infty^2(q^2;q^3)_\infty^2}\\&=\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}.\end{aligned}$$

The last equality in (3.3.7) follows from the following general considerations. For arbitrary positive integers j and k,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^{kn}}{1-q^{jn+k}}&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q^{kn+km+jnm}\\&=\sum_{n=0}^{\infty}\left(\sum_{m>n}+\sum_{0\le m\le n}\right)q^{kn+km+jnm}\\&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}q^{kn+k(m+n+1)+jn(m+n+1)}+\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}q^{k(n+m)+km+j(n+m)m}\\&=\sum_{n=0}^{\infty}\frac{q^{jn^2+(j+2k)n+k}}{1-q^{jn+k}}+\sum_{m=0}^{\infty}\frac{q^{jm^2+2km}}{1-q^{jm+k}}\\&=\sum_{n=0}^{\infty}\frac{q^{jn^2+2kn}(1+q^{jn+k})}{1-q^{jn+k}},\end{aligned}$$

where in the latter sum in the penultimate line we replaced m by n.

The case $j=3$, $k=1$ in the calculation above immediately yields

$$\sum_{n=0}^{\infty}\frac{q^n}{1-q^{3n+1}}=\sum_{n=0}^{\infty}\frac{q^{3n^2+2n}(1+q^{3n+1})}{1-q^{3n+1}}, \tag{3.3.9}$$

while the case $j=3$, $k=2$ yields, after multiplying both sides by q,

$$\sum_{n=0}^{\infty}\frac{q^{2n+1}}{1-q^{3n+2}}=\sum_{n=0}^{\infty}\frac{q^{3n^2+4n+1}(1+q^{3n+2})}{1-q^{3n+2}}. \tag{3.3.10}$$

Thus, substituting the right sides of (3.3.9) and (3.3.10) into the first line of (3.3.7), we immediately establish the final portion of Entry 3.3.4. □

The right-hand side (3.3.8) can be written in a more symmetric form as a bilateral series

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} = q^{-1/3} \sum_{k=-\infty}^{\infty} \frac{1+q^{3k+1}}{1-q^{3k+1}} q^{(3k+1)^2/3}.$$

Moreover, the previous right-hand side (3.3.7) may be written in the symmetric form

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} = q^{-1/3} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \left\{ q^{(3j+1)(3k+1)/3} - q^{(3j+2)(3k+2)/3} \right\}. \tag{3.3.11}$$

If Clausen's transformation [54, p. 113] is applied to the double series in (3.3.11), we find that

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} = q^{-1/3} \sum_{n=-\infty}^{\infty} \left(\frac{n}{3}\right) \frac{q^{n^2/3}}{1-q^n},$$

where $\left(\frac{n}{3}\right)$ denotes the Legendre symbol. The function on the right-hand sides of (3.3.7), (3.3.8), and (3.3.11) is equal to $\frac{1}{3}q^{-1/3}c(q)$, where $c(q)$ is one of the cubic theta functions of J.M. and P.B. Borwein [90] defined by

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}.$$

For a proof of this remark, see Berndt's book [57, p. 109, Lemma 5.1, equation (5.5)]. The function $c(q)$ is crucial in the development of Ramanujan's cubic theory of elliptic functions set out by Ramanujan in his second notebook [243, pp. 257–262] and first established by Berndt, Bhargava, and F.G. Garvan [60], [57, Chapter 33]. Undoubtedly, Ramanujan was the first to prove (3.3.7), but the first proof in print was probably that of Fine in his book [149, p. 79, equation (32.35)]. Another proof has been given by L.-C. Shen [258].

The identity (3.3.7) can be greatly generalized. For $|q| < |x| < 1$ and any number y,

$$\frac{(xy;q)_\infty (q/(xy);q)_\infty (q;q)_\infty^2}{(x;q)_\infty (q/x;q)_\infty (y;q)_\infty (q/y;q)_\infty} = \sum_{n=-\infty}^{\infty} \frac{x^n}{1-yq^n}. \tag{3.3.12}$$

If we replace q by q^3 in (3.3.12) and then set $x = y = q$, we readily find that (3.3.12) reduces to (3.3.7). We are uncertain who first proved (3.3.12). The earliest reference for (3.3.12) known to us is by L. Kronecker [196], [197, pp. 309–318] ; see A. Weil's monograph [282, pp. 70–71] for an account of Kronecker's proof. Another proof can be found in K. Venkatachaliengar's monograph [272, p. 37]. A short proof can be found in S.H. Chan's thesis [117] and paper [118]. Special cases of (3.3.12) are useful in deriving Lambert series identities arising in the number of representations of an integer as a sum of squares or triangular numbers [115], [139], [258]. Its relation to Chapter 16 of Ramanujan's original second notebook is found in [27].

A further generalization of (3.3.7) and (3.3.12) involving one additional variable was discovered by S. McCullough and Shen [220] and by Andrews, R. Lewis, and Liu [37, Theorem 1]. A simpler proof was found by S.H. Chan [117], [118]. This identity is also equivalent to an identity for Lambert series found by K. Venkatachaliengar [272, p. 37] and elaborated upon in more detail by S. Cooper [139]. However, the aforementioned generalization of these authors is actually equivalent to an identity for the Weierstrass σ- and ζ-functions, which was given by G. Halphen [173]. This identity,

$$\begin{aligned}\frac{\sigma(a+b)\sigma(a-b)}{\sigma(a+c)\sigma(a-c)\sigma(b+c)\sigma(b-c)}&\\ =\frac{1}{\sigma(2c)}\left(\zeta(a+c)-\zeta(a-c)+\zeta(b-c)-\zeta(b+c)\right),&\end{aligned}$$

can be deduced from an exercise in Whittaker and Watson's text [283, p. 451, Exercise 5] that is originally due to C. Weierstrass, namely,

$$\begin{aligned}\sigma(a+b)\sigma(a-b)\sigma(c+d)\sigma(c-d)+\sigma(b+c)\sigma(b-c)\sigma(a+d)\sigma(a-d)&\\ +\sigma(c+a)\sigma(c-a)\sigma(b+d)\sigma(b-d)=0,&\end{aligned}$$

by dividing both sides by $c-d$ and letting $d\to c$.

The final entry for this section is, as we shall see, merely a transformed version of identity (3.1.4).

Entry 3.3.5 (p. 1). *For* $|q/a|<|1/b|<1$ *and* $abc\neq 0$,

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(-q/c;q)_n}{(-q/a;q)_n(-q/b;q)_n}\left(\frac{c}{ab}\right)^n q^{n(n-1)/2}\\ &+(1+a^{-1})(1+b^{-1})\sum_{n=1}^{\infty}\frac{(-q/c;q)_{n-1}}{(aq/c;q)_n(bq/c;q)_n}\left(\frac{ab}{c}\right)^n q^{n(n+1)/2}\\ &=\frac{(-q/c;q)_\infty}{(aq/c,bq/c,-q/a,-q/b;q)_\infty}\sum_{n=0}^{\infty}\left(\frac{a^nb^n}{c^n}+\frac{c^{n+1}}{a^{n+1}b^{n+1}}\right)q^{n(n+1)/2}.\end{aligned}\tag{3.3.13}$$

Proof. We begin by noting that, by the Jacobi triple product identity (1.4.8),

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{a^nb^n}{c^n}+\frac{c^{n+1}}{a^{n+1}b^{n+1}}\right)q^{n(n+1)/2}&=\sum_{n=-\infty}^{\infty}\left(\frac{ab}{c}\right)^n q^{n(n+1)/2}\\ &=(-abq/c;q)_\infty(-c/(ab);q)_\infty(q;q)_\infty.\end{aligned}$$

Therefore, the right-hand side of (3.3.13) is, in fact, equal to

$$\frac{(-q/c, q, -abq/c, -c/(ab); q)_\infty}{(aq/c, bq/c, -q/a, -q/b; q)_\infty}. \tag{3.3.14}$$

Next, we apply (2.1.1) to the first sum on the left-hand side of (3.3.13) with a, b, c, d, and e replaced respectively by q, $-q/c$, $-1/\tau$, $-q/a$, and $-q/b$, and then let τ tend to 0 to obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-q/c;q)_n}{(-q/a;q)_n(-q/b;q)_n}\left(\frac{c}{ab}\right)^n q^{n(n-1)/2} &= \lim_{\tau\to 0} {}_3\phi_2\left(\begin{matrix} q, -q/c, -1/\tau \\ -q/a, -q/b \end{matrix}; q, \frac{c\tau}{ab}\right) \\ &= (1+b^{-1})\sum_{n=0}^{\infty} \frac{(-1)^n(c/a;q)_n}{(-q/a;q)_n} b^{-n}.\end{aligned} \tag{3.3.15}$$

Applying (2.1.1) to the second sum on the left-hand side of (3.3.13) with a, b, c, d, and e replaced respectively by q, $-q^2/\tau$, $-q/c$, aq^2/c, and bq^2/c and then letting τ tend to 0, we deduce that

$$\begin{aligned}&\sum_{n=1}^{\infty} \frac{(-q/c;q)_{n-1}}{(aq/c;q)_n(bq/c;q)_n}\left(\frac{ab}{c}\right)^n q^{n(n+1)/2} \\ &\quad = \frac{abq/c}{(1-aq/c)(1-bq/c)} \lim_{\tau\to 0} {}_3\phi_2\left(\begin{matrix} q, -q^2/\tau, -q/c \\ aq^2/c, bq^2/c \end{matrix}; q, \frac{\tau ab}{c}\right) \\ &\quad = \frac{abq}{c(1-aq/c)} \sum_{n=0}^{\infty} \frac{(-aq;q)_n}{(aq^2/c;q)_n}\left(\frac{bq}{c}\right)^n.\end{aligned} \tag{3.3.16}$$

Hence, using (3.3.14)–(3.3.16) in (3.3.13), we see that we can reduce the proof of Entry 3.3.5 to proving the assertion

$$\begin{aligned}&(1+b^{-1})\sum_{n=0}^{\infty} \frac{(-1)^n(c/a;q)_n}{(-q/a;q)_n} b^{-n} \\ &\qquad + \frac{abq(1+a^{-1})(1+b^{-1})}{c(1-aq/c)} \sum_{n=0}^{\infty} \frac{(-aq;q)_n}{(aq^2/c;q)_n}\left(\frac{bq}{c}\right)^n \\ &\quad = \frac{(-q/c, q, -abq/c, -c/(ab); q)_\infty}{(aq/c, bq/c, -q/a, -q/b; q)_\infty}.\end{aligned} \tag{3.3.17}$$

By (3.1.1), we find that

$$\begin{aligned}\sum_{n=-1}^{-\infty} \frac{(-1)^n(c/a;q)_n}{(-q/a;q)_n} b^{-n} &= \sum_{n=1}^{\infty} \frac{(-a;q)_n}{(aq/c;q)_n}\left(\frac{bq}{c}\right)^n \\ &= \frac{(1+a)}{(1-aq/c)} \sum_{n=0}^{\infty} \frac{(-aq;q)_n}{(aq^2/c;q)_n}\left(\frac{bq}{c}\right)^{n+1}.\end{aligned} \tag{3.3.18}$$

Substituting (3.3.18) into (3.3.17) and then multiplying the resulting identity by $1/(1+b^{-1})$, we find that (3.3.17) reduces to

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n (c/a;q)_n}{(-q/a;q)_n} b^{-n} = \frac{(-q/c, q, -abq/c, -c/(ab);q)_\infty}{(aq/c, bq/c, -q/a, -1/b;q)_\infty}, \tag{3.3.19}$$

and this is merely (3.1.4) with a, b, and z replaced by c/a, $-q/a$, and $-1/b$, respectively.

We have ignored needed conditions on a, b, and c in our proof. However, if we translate the hypotheses necessary for the validity of (3.1.4) into those needed for Entry 3.3.5 to hold, we arrive at the hypotheses given for Entry 3.3.5. □

3.4 The ${}_2\psi_2$ Identities

Entry 3.4.1 (pp. 6, 14). *Let a denote any complex number. Recall that $f(a,b)$ and $\psi(q)$ are defined by* (1.4.8) *and* (1.4.10), *respectively. Then*

$$\sum_{n=0}^{\infty} \frac{(-aq;q^2)_n}{(aq;q)_{2n}} a^n q^{n^2} + \sum_{n=1}^{\infty} \frac{(1/a;q)_{2n}}{(-q/a;q^2)_n} q^n$$
$$= \frac{\psi(q)\left(f(a^3q^2, a^{-3}q^4) + a\, f(a^3q^4, a^{-3}q^2)\right)}{(aq;q)_\infty(-q/a;q^2)_\infty f(a, q^2/a)}. \tag{3.4.1}$$

Proof. We begin by observing that, by (3.1.1),

$$\frac{(-aq;q^2)_{-n}}{(aq;q)_{-2n}} a^{-n} q^{(-n)^2} = \frac{(q/aq;q)_{2n}(aq)^{-n} q^{n(n+1)}}{(q^2/(-aq);q^2)_n(-aq)^{-2n} q^{2n(2n+1)/2}} a^{-n} q^{n^2}$$
$$= \frac{(1/a;q)_{2n}}{(-q/a;q^2)_n} q^n.$$

Now use (3.2.4) with q replaced by q^2, a replaced by a^2, and c, d, e, and f replaced by aq, a, $-q/\tau$, and $-aq$, respectively. Using also the calculation above and letting τ tend to 0, we find that

$$\sum_{n=0}^{\infty} \frac{(-aq;q^2)_n}{(aq;q)_{2n}} a^n q^{n^2} + \sum_{n=1}^{\infty} \frac{(1/a;q)_{2n}}{(-q/a;q^2)_n} q^n$$
$$= \sum_{n=-\infty}^{\infty} \frac{(-aq;q^2)_n}{(aq;q)_{2n}} a^n q^{n^2}$$
$$= \lim_{\tau\to 0} {}_2\psi_2\left(\begin{matrix} -q/\tau, & -aq \\ aq, & aq^2 \end{matrix}; q^2, a\tau\right)$$
$$= \frac{(q/a;q^2)_\infty(q^2/a;q^2)_\infty(-aq;q^2)_\infty}{(a^2q^2;q^2)_\infty(q^2/a^2;q^2)_\infty(q;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{1+aq^{2n}}{1+a} a^{3n} q^{n(3n-1)}$$
$$= \frac{(q/a;q)_\infty(-aq;q^2)_\infty}{(aq;q)_\infty(-a;q)_\infty(q/a;q)_\infty(-q/a;q)_\infty(q;q^2)_\infty} \sum_{n=-\infty}^{\infty} (1+aq^{2n}) a^{3n} q^{n(3n-1)}$$

$$= \frac{(q^2;q^2)_\infty \left(f(a^3q^2,a^{-3}q^4)+a\,f(a^3q^4,a^{-3}q^2)\right)}{(aq;q)_\infty(-a;q^2)_\infty(-q/a;q^2)_\infty(-q^2/a;q^2)_\infty(q;q^2)_\infty(q^2;q^2)_\infty}$$
$$= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\frac{\left(f(a^3q^2,a^{-3}q^4)+a\,f(a^3q^4,a^{-3}q^2)\right)}{(aq;q)_\infty(-q/a;q^2)_\infty}$$
$$\times\frac{1}{(-a;q^2)_\infty(-q^2/a;q^2)_\infty(q^2;q^2)_\infty}.$$

Applying (1.4.10) and (1.4.8) on the far right side above, we arrive at the right-hand side of (3.4.1). □

The next entry is a natural companion to Entry 3.4.1.

Entry 3.4.2 (p. 14). *Let a denote any complex number. Recall that $f(a,b)$ and $\psi(q)$ are defined by* (1.4.8) *and* (1.4.10), *respectively. Then*

$$\sum_{n=0}^{\infty}\frac{(-aq;q^2)_n}{(aq;q)_{2n+1}}a^{n+1}q^{(n+1)^2}-\sum_{n=1}^{\infty}\frac{(1/a;q)_{2n-1}}{(-q/a;q^2)_n}q^n$$
$$=\frac{q\psi(q)f(q^6/a^3,a^3)}{a(aq;q)_\infty(-q/a;q^2)_\infty f(q^2/a,a)}. \tag{3.4.2}$$

Proof. We begin by observing that, by (3.1.1),

$$\frac{(-aq;q^2)_{-n}}{(aq;q)_{-2n+1}}a^{-n+1}q^{(-n+1)^2}=-\frac{(1/a;q)_{2n-1}}{(-q/a;q^2)_n}q^n.$$

So we see that

$$\sum_{n=0}^{\infty}\frac{(-aq;q^2)_n}{(aq;q)_{2n+1}}a^{n+1}q^{(n+1)^2}-\sum_{n=1}^{\infty}\frac{(1/a;q)_{2n-1}}{(-q/a;q^2)_n}q^n$$
$$=\sum_{n=-\infty}^{\infty}\frac{(-aq;q^2)_n}{(aq;q)_{2n+1}}a^{n+1}q^{(n+1)^2}. \tag{3.4.3}$$

Apply (3.2.4) with q replaced by q^2, a replaced by a^2q^2, and then c, d, e, and f replaced by aq^2, aq, $-1/\tau$, and $-aq$, respectively. Letting τ tend to 0, we find that

$$\sum_{n=-\infty}^{\infty}\frac{(-aq;q^2)_n}{(aq;q)_{2n+1}}a^{n+1}q^{(n+1)^2}$$
$$=\frac{aq}{1-aq}\lim_{\tau\to0}{}_2\psi_2\left(\begin{matrix}-1/\tau, & -aq\\ aq^2, & aq^3\end{matrix};q^2,a\tau q^3\right)$$
$$=\frac{aq}{1-aq}\frac{(1/a,q/a,-aq^3;q^2)_\infty}{(a^2q^4,1/a^2,q;q^2)_\infty}\sum_{n=-\infty}^{\infty}q^{3n^2+3n}a^{3n}. \tag{3.4.4}$$

Replacing the index of summation n by $-n-1$, simplifying the products, and using the Jacobi triple product identity (1.4.8) twice and (1.4.10) once, we find from (3.4.4) that

$$\begin{aligned}
&\sum_{n=-\infty}^{\infty} \frac{(-aq;q^2)_n}{(aq;q)_{2n+1}} a^{n+1} q^{(n+1)^2} \\
&\quad = \frac{aq(1/a;q)_\infty(-aq;q^2)_\infty}{(aq;q)_\infty(-aq;q)_\infty(1/a;q)_\infty(-1/a;q)_\infty(q;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{3n^2+3n} a^{-3n-3} \\
&\quad = \frac{a^{-2}q(q^2;q^2)_\infty}{(q;q^2)_\infty} \frac{f(q^6/a^3,a^3)}{(aq;q)_\infty(-q/a;q^2)_\infty} \frac{1}{(-aq^2;q^2)_\infty(-1/a;q^2)_\infty(q^2;q^2)_\infty} \\
&\quad = \frac{a^{-2}q\psi(q)f(q^6/a^3,a^3)}{(aq;q)_\infty(-q/a;q^2)_\infty f(1/a,aq^2)}. \qquad (3.4.5)
\end{aligned}$$

Use the identity $a\,f(1/a,aq^2) = f(q^2/a,a)$, which is a consequence of (1.4.13), in (3.4.5). Then substitute (3.4.5) into (3.4.3). We thus obtain the right-hand side of (3.4.2) to complete the proof. ☐

The next entry is an immediate corollary of Entry 3.4.1.

Entry 3.4.3 (pp. 6, 16).

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1)^n(q;q^2)_n}{(-q;q)_{2n}} q^{n^2} + 2\sum_{n=1}^{\infty} \frac{(-q;q)_{2n-1}}{(q;q^2)_n} q^n \\
&\quad = \frac{(-q;q^2)_\infty}{\varphi(-q^2)} \left(1 + 6\sum_{n=0}^{\infty} \frac{q^{6n+2}}{1-q^{6n+2}} - 6\sum_{n=0}^{\infty} \frac{q^{6n+4}}{1-q^{6n+4}}\right). \qquad (3.4.6)
\end{aligned}$$

Proof. The left-hand side of (3.4.6) is the left-hand side of Entry 3.4.1 at $a=-1$. On the other hand, for the right-hand side of (3.4.1), by (1.4.8),

$$\begin{aligned}
&\lim_{a\to-1} \frac{\psi(q)\left(f(a^3q^2,a^{-3}q^4) + a\,f(a^3q^4,a^{-3}q^2)\right)}{(aq;q)_\infty(-q/a;q^2)_\infty f(a,q^2/a)} \\
&\quad = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty(-q;q)_\infty(q;q^2)_\infty} \\
&\qquad \times \lim_{a\to-1} \frac{(-a^3q^2,-a^{-3}q^4,q^6;q^6)_\infty + a(-a^3q^4,-a^{-3}q^2,q^6;q^6)_\infty}{(-a,-a^{-1}q^2,q^2;q^2)_\infty} \\
&\quad = \frac{1}{(q;q^2)_\infty(q^2;q^2)_\infty^2} \\
&\qquad \times \lim_{a\to-1} \frac{(-a^3q^2,-a^{-3}q^4,q^6;q^6)_\infty + a(-a^3q^4,-a^{-3}q^2,q^6;q^6)_\infty}{1+a} \\
&\quad = \frac{(q^6;q^6)_\infty}{(q;q^2)_\infty(q^2;q^2)_\infty^2}
\end{aligned}$$

$$
\begin{aligned}
&\times \lim_{a\to -1}(q^2;q^6)_\infty(q^4;q^6)_\infty\left\{\frac{d}{da}\log\{(-a^3q^2;q^6)_\infty(-a^{-3}q^4;q^6)_\infty\}\right.\\
&\left.+\frac{d}{da}a+a\frac{d}{da}\log\{(-a^3q^4;q^6)_\infty(-a^{-3}q^2;q^6)_\infty\}\right\}\\
&=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty(q^2;q^2)_\infty^2}\left(1+\sum_{n=0}^\infty\frac{6q^{6n+2}}{1-q^{6n+2}}-\sum_{n=0}^\infty\frac{6q^{6n+4}}{1-q^{6n+4}}\right)\\
&=\frac{(-q;q^2)_\infty(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\left(1+\sum_{n=0}^\infty\frac{6q^{6n+2}}{1-q^{6n+2}}-\sum_{n=0}^\infty\frac{6q^{6n+4}}{1-q^{6n+4}}\right),
\end{aligned}
$$

by Euler's identity. This last expression is equivalent to the right-hand side of (3.4.6) upon an application of (1.4.9). □

Entry 3.4.4 (p. 14).

$$\sum_{n=0}^\infty\frac{(-q;q^2)_n}{(q;q)_{2n+1}}q^{(n+1)^2}=\frac{q\psi(q^6)}{\psi(-q)}. \tag{3.4.7}$$

Proof. Set $a=1$ in Entry 3.4.2. The resulting left-hand side is the left-hand side of (3.4.7). The right-hand side of (3.4.2), with applications of (1.4.10) and (1.4.8), becomes

$$
\begin{aligned}
\frac{q\psi(q)f(q^6,1)}{(q;q)_\infty(-q;q^2)_\infty f(q^2,1)}&=\frac{q\psi(q)2\psi(q^6)}{(q;q)_\infty(-q;q^2)_\infty 2\psi(q^2)}\\
&=\frac{q(q^2;q^2)_\infty(q^2;q^4)_\infty\psi(q^6)}{(q;q)_\infty(-q;q^2)_\infty(q;q^2)_\infty(q^4;q^4)_\infty}\\
&=\frac{q\psi(q^6)}{(q;q^2)_\infty(q^4;q^4)_\infty}\\
&=\frac{q\psi(q^6)}{(q;q^4)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty}\\
&=\frac{q\psi(q^6)}{\psi(-q)},
\end{aligned}
$$

and this completes the proof. □

Entry 3.4.4 is due to Slater [262, equation (50)].

Entry 3.4.5 (p. 14).

$$\sum_{n=0}^\infty\frac{(-q;q^2)_n}{(q;q)_{2n+1}}q^n=\frac{\psi(-q^3)}{\varphi(-q)}. \tag{3.4.8}$$

Proof. Apply (1.2.10) with q replaced by q^2 and $b=-q$, $c=q^3$, and $t=q$. Letting a tend to 0, we find that

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q;q)_{2n+1}} q^n = \frac{1}{1-q} \sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^2;q^2)_n (q^3;q^2)_n} q^n$$
$$= \frac{1}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n}{(q;q)_{2n+1}} q^{n^2+n}. \tag{3.4.9}$$

We apply Entry 3.4.1 with $a = q$, employ the identity $qf(q^7, q^{-1}) = f(q, q^5)$, deducible from (1.4.13), and use (3.4.9), (1.4.8), (1.4.10), and (1.4.9) twice to deduce that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q;q)_{2n+1}} q^n &= \frac{1}{(q;q^2)_\infty} \frac{\psi(q)\big(f(q^5,q) + qf(q^7,q^{-1})\big)}{(q;q)_\infty (-1;q^2)_\infty \varphi(q)} \\
&= \frac{(q^2;q^2)_\infty 2f(q,q^5)}{(q;q^2)^2_\infty (q;q)_\infty 2(-q^2;q^2)_\infty (-q;q^2)^2_\infty (q^2;q^2)_\infty} \\
&= \frac{(-q;q^6)_\infty (-q^5;q^6)_\infty (q^6;q^6)_\infty}{(q^2;q^4)_\infty (q;q)_\infty} \\
&= \frac{(-q;q^2)_\infty (q^6;q^6)_\infty}{(q^2;q^4)_\infty (q;q)_\infty (-q^3;q^6)_\infty} \\
&= \frac{(-q;q^2)_\infty (q^3;q^6)_\infty (q^6;q^6)_\infty}{(q^2;q^4)_\infty (q;q)_\infty (q^6;q^{12})_\infty} \\
&= \frac{(-q;q^2)_\infty (q^3;q^{12})_\infty (q^6;q^{12})_\infty (q^9;q^{12})_\infty (q^{12};q^{12})_\infty}{(q^2;q^4)_\infty (q;q)_\infty (q^6;q^{12})_\infty} \\
&= \frac{f(-q^3,-q^9)}{(q;q^2)_\infty (q;q)_\infty} \\
&= \frac{\psi(-q^3)}{\varphi(-q)},
\end{aligned}$$

as desired. □

Entry 3.4.6 (p. 22).

$$\sum_{n=1}^{\infty} \frac{q^{2n^2-n}}{(q;q^2)_n} + \sum_{n=0}^{\infty} (-1)^n (q;q^2)_n q^{n^2+n} = (q^2;q^2)^2_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)^2_n}. \tag{3.4.10}$$

Proof. We begin by noting that, by (3.1.1),

$$(-1)^{-n} (q;q^2)_{-n} q^{(-n)^2-n} = \frac{q^{2n^2-n}}{(q;q^2)_n}.$$

Using first the equality above, employing secondly (3.2.3) with q replaced by q^2 and with $\alpha = q/\tau$, $\beta = q$, $\delta = \gamma = \tau$, and $z = q\tau$, thirdly letting τ tend to 0, and fourthly applying Entry 1.5.3, we find that

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{q^{2n^2-n}}{(q;q^2)_n}+\sum_{n=0}^{\infty}(-1)^n(q;q^2)_nq^{n^2+n} &= \sum_{n=-\infty}^{\infty}(-1)^n(q;q^2)_nq^{n^2+n}\\
&= \lim_{\tau\to 0}{}_2\psi_2\left(\begin{matrix}q/\tau,\ q\\ \tau,\ \ \tau\end{matrix};q^2,q\tau\right)\\
&= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\lim_{\tau\to 0}{}_2\psi_2\left(\begin{matrix}q^3/\tau,\ q^3/\tau\\ q^2,\ \ q^2\tau\end{matrix};q^2,\tau^2/q^3\right)\\
&= (-q;q)_\infty(q^2;q^2)_\infty\sum_{n=0}^{\infty}\frac{q^{2n^2+n}}{(q^2;q^2)_n}\\
&= \frac{(-q;q)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n+1)/2}}{(q^2;q^2)_n}\\
&= (q^2;q^2)_\infty^2\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q;q)_n^2},
\end{aligned}$$

where we invoked Entry 1.5.3, and next applied (1.2.1) with $h=1$, $a=q/t$, and $c=-q$, and then let b and t tend to 0. This completes the proof. □

Entry 3.4.7 (p. 15). *For $a,b\neq 0$,*

$$\sum_{n=0}^{\infty}\frac{a^{-n-1}b^{-n}}{(-1/a;q)_{n+1}(-q/b;q)_n}q^{n^2}+\sum_{n=1}^{\infty}(-aq;q)_{n-1}(-b;q)_nq^n \tag{3.4.11}$$
$$=\frac{(-aq;q)_\infty}{(q;q)_\infty(-q/b;q)_\infty}\left(\sum_{n=0}^{\infty}\frac{b^nq^{n(n+1)/2}}{1+aq^n}+\frac{1}{a}\sum_{n=1}^{\infty}\frac{b^{-n}q^{n(n+1)/2}}{1+q^n/a}\right).$$

Proof. In light of the fact that, by (3.1.1),

$$\frac{a^{-(-n)-1}b^nq^{(-n)^2}}{(-1/a;q)_{-n+1}(-q/b;q)_{-n}}=(-aq;q)_{n-1}(-b;q)_nq^n,$$

and

$$\frac{b^{-n}q^{(-n)(-n+1)/2}}{1+aq^{(-n)}}=\frac{a^{-1}b^{-n}q^{n(n+1)/2}}{1+q^n/a},$$

we see that we may rewrite (3.4.11) as

$$\sum_{n=-\infty}^{\infty}\frac{a^{-n-1}b^{-n}}{(-1/a;q)_{n+1}(-q/b;q)_n}q^{n^2}=\frac{(-aq;q)_\infty}{a(q;q)_\infty(-q/b;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{b^{-n}q^{n(n+1)/2}}{1+q^n/a}. \tag{3.4.12}$$

This last identity is, after simplification, the special case of (3.2.2) in which we let $\alpha=\beta=1/\tau$, $\gamma=-q/a$, $\delta=-q/b$, and $z=q\tau^2/(ab)$, multiply both sides of (3.2.2) by $1/(1+a)$, and then let τ tend to 0. □

Entry 3.4.8 (p. 15). *For $a \neq 0$,*

$$\sum_{n=1}^{\infty}(-1;q)_n(-aq;q)_{n-1}q^n + \frac{1}{1+a}\sum_{n=0}^{\infty}\frac{a^{-n}q^{n^2}}{(-q;q)_n(-q/a;q)_n}$$
$$= \frac{(-aq;q)_\infty}{(q^2;q^2)_\infty}\left(\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{1+aq^n} + \frac{1}{a}\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}}{1+q^n/a}\right).$$

Proof. Set $b = 1$ in Entry 3.4.7. □

3.5 Identities Arising from the Quintuple Product Identity

Entry 3.5.1 (p. 48).

$$(q^2;q^2)_\infty \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \sum_{n=-\infty}^{\infty}(-1)^n q^{15n^2+2n}(1+q^{6n+1}). \tag{3.5.1}$$

Proof. By the first Rogers–Ramanujan identity (Entry 3.2.2 of Part I [31, p. 87] or (4.1.1) in this volume), we find that

$$\begin{aligned}
(q^2;q^2)_\infty \sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} &= \frac{(q^2;q^2)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} \\
&= \frac{(q^2;q^2)_\infty(-q;q^5)_\infty(-q^4;q^5)_\infty}{(q^2;q^{10})_\infty(q^8;q^{10})_\infty} \\
&= (q^{10};q^{10})_\infty(-q;q^5)_\infty(-q^4;q^5)_\infty(q^4;q^{10})_\infty(q^6;q^{10})_\infty \\
&= (q^{10};q^{10})_\infty(-q;q^{10})_\infty(-q^9;q^{10})_\infty(q^8;q^{20})_\infty(q^{12};q^{20})_\infty \\
&= \sum_{n=-\infty}^{\infty}(-1)^n q^{15n^2+2n}(1+q^{6n+1}),
\end{aligned} \tag{3.5.2}$$

where in the last equality we used (3.1.2) with q replaced by q^{10} and z replaced by q. □

Entry 3.5.2 (p. 48).

$$(q^2;q^2)_\infty \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n} = \sum_{n=-\infty}^{\infty}(-1)^n q^{15n^2+4n}(1+q^{10n+3}). \tag{3.5.3}$$

Proof. By the second Rogers–Ramanujan identity (Entry 3.2.2 of Part I [31, p. 87] or (4.1.2) in this book), we find that

$$
\begin{aligned}
(q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} &= \frac{(q^2;q^2)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty} \\
&= \frac{(q^2;q^2)_\infty (-q^2;q^5)_\infty (-q^3;q^5)_\infty}{(q^4;q^{10})_\infty (q^6;q^{10})_\infty} \\
&= (q^{10};q^{10})_\infty (-q^2;q^5)_\infty (-q^3;q^5)_\infty (q^2;q^{10})_\infty (q^8;q^{10})_\infty \\
&= (q^{10};q^{10})_\infty (-q^7;q^{10})_\infty (-q^3;q^{10})_\infty (q^4;q^{20})_\infty (q^{16};q^{20})_\infty \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{15n^2+4n}(1+q^{10n+3}), \qquad (3.5.4)
\end{aligned}
$$

where in the last equality we used (3.1.2) with q replaced by q^{10} and z replaced by q^3. □

Entry 3.5.3 (p. 36). *We have*

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+7n+2)/2}}{(-q^3;q^5)_{n+1}} &= \frac{1}{2}\sum_{n=0}^{\infty} q^{n(15n+7)/2}(1-q^{8n+4}) \\
&\quad + \frac{1}{2}\sum_{n=0}^{\infty} q^{(15n^2+13n+2)/2}(1-q^{2n+1}) - \frac{f(-q^2,-q^8)}{2(-q;q)_\infty}, \qquad (3.5.5)
\end{aligned}
$$

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+3n)/2}}{(-q^2;q^5)_{n+1}} &= \frac{1}{2}\sum_{n=0}^{\infty} q^{n(15n+7)/2}(1-q^{8n+4}) \\
&\quad + \frac{1}{2}\sum_{n=0}^{\infty} q^{(15n^2+13n+2)/2}(1-q^{2n+1}) + \frac{f(-q^2,-q^8)}{2(-q;q)_\infty}. \qquad (3.5.6)
\end{aligned}
$$

We have combined these two results into a single entry because it is easiest to prove them simultaneously.

Proof. We propose to prove the following two formulas, from which the desired results follow by addition and subtraction:

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+3n)/2}}{(-q^2;q^5)_{n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+7n+2)/2}}{(-q^3;q^5)_{n+1}} \qquad (3.5.7)\\
&\quad = \sum_{n=0}^{\infty} q^{n(15n+7)/2}(1-q^{8n+4}) + \sum_{n=0}^{\infty} q^{(15n^2+13n+2)/2}(1-q^{2n+1}),
\end{aligned}
$$

$$
\sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+3n)/2}}{(-q^2;q^5)_{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+7n+2)/2}}{(-q^3;q^5)_{n+1}} = \frac{f(-q^2,-q^8)}{(-q;q)_\infty}. \qquad (3.5.8)
$$

To prove these last two results, we first apply the Rogers–Fine identity (1.7.1) with q replaced by q^5, $\alpha = q^6/\tau$, and $\beta = -q^8$. Let τ tend to 0 and multiply both sides by $q/(1+q^3)$. Accordingly,

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+7n+2)/2}}{(-q^3;q^5)_{n+1}}$$
$$= \frac{q}{1+q^3} \lim_{\tau\to 0} \sum_{n=0}^{\infty} \frac{(q^6/\tau;q^5)_n(-q^3;q^5)_n(-1)^n q^{8n}\tau^n q^{5n^2-5n}(1-q^{10n+6})}{(-q^8;q^5)_n(\tau;q^5)_{n+1}}$$
$$= \sum_{n=0}^{\infty} q^{(15n^2+13n+2)/2}(1-q^{5n+3}). \tag{3.5.9}$$

Also, applying (1.7.1) with q replaced by q^5, $\alpha = q^4/\tau$, and $\beta = -q^7$, letting τ tend to 0, and then multiplying both sides by $1/(1+q^2)$, we deduce that

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+3n)/2}}{(-q^2;q^5)_{n+1}} = \sum_{n=0}^{\infty} q^{(15n^2+7n)/2}(1-q^{5n+2}). \tag{3.5.10}$$

Hence, if we add (3.5.9) and (3.5.10), we see that (3.5.7) follows from

$$\sum_{n=0}^{\infty} q^{(15n^2+7n)/2}(1-q^{5n+2}) + \sum_{n=0}^{\infty} q^{(15n^2+13n+2)/2}(1-q^{5n+3})$$
$$= \sum_{n=0}^{\infty} \left(q^{(15n^2+7n)/2} - q^{(15n^2+17n+4)/2} + q^{(15n^2+13n+2)/2} - q^{(15n^2+23n+8)/2}\right)$$
$$= \sum_{n=0}^{\infty} q^{(15n^2+7n)/2}(1-q^{8n+4}) + \sum_{n=0}^{\infty} q^{(15n^2+13n+2)/2}(1-q^{2n+1}).$$

To obtain (3.5.8), we see that we must subtract (3.5.9) from (3.5.10). Thus, we need to determine

$$\sum_{n=0}^{\infty} q^{(15n^2+7n)/2}(1-q^{5n+2}) - \sum_{n=0}^{\infty} q^{(15n^2+13n+2)/2}(1-q^{5n+3})$$
$$= \sum_{n=0}^{\infty} \Big(q^{(15n^2+7n)/2} - q^{(15n^2+17n+4)/2}$$
$$- q^{(15(-n-1)^2+17(-n-1)+4)/2} + q^{(15(-n-1)^2+7(-n-1))/2}\Big)$$
$$= \sum_{n=-\infty}^{\infty} q^{(15n^2+7n)/2} - \sum_{n=-\infty}^{\infty} q^{(15n^2+17n+4)/2}$$
$$= \sum_{n=-\infty}^{\infty} q^{(15n^2+7n)/2}(1-q^{5n+2}). \tag{3.5.11}$$

Applying (3.1.2) with q replaced by q^5 and z replaced by $-q^2$, we find that

$$\sum_{n=-\infty}^{\infty} q^{(15n^2+7n)/2}(1-q^{5n+2}) = (q^2;q^5)_\infty(q^3;q^5)_\infty(q^5;q^5)_\infty(q;q^{10})_\infty(q^9;q^{10})_\infty$$

$$
\begin{aligned}
&= (q^2;q^{10})_\infty (q^8;q^{10})_\infty (q^{10};q^{10})_\infty (q;q^2)_\infty \\
&= \frac{f(-q^2,-q^8)}{(-q;q)_\infty}, \qquad (3.5.12)
\end{aligned}
$$

where we used (1.4.8) in the last equality. This proves (3.5.8). □

The following three entries are, respectively, formulas (117), (118), and (119) in Slater's paper [262], but with q replaced by $-q$. The three formulas are also related, respectively, to formulas (80), (81), and (82) in Slater's paper [262], which can be found in Rogers's paper [248, p. 331, equation (1), lines 2, 1, and 3, resp.]. Although results from Chapter 1 are required in our proofs, in light of the fact that the quintuple product identity plays a central role, we have placed these entries here.

Entry 3.5.4 (p. 10). *Recalling that* $f(a,b)$ *is defined in* (1.4.8)*, we have*

$$
\sum_{n=0}^\infty \frac{(-1)^n q^{n^2}}{(q^4;q^4)_n(-q;q^2)_n} = \frac{f(q^{10},q^{11}) - qf(q^4,q^{17})}{(q^2;q^2)_\infty} = \frac{(q^7;q^7)_\infty f(-q^2,-q^5)}{(q^2;q^2)_\infty f(q,q^6)}. \qquad (3.5.13)
$$

Proof. We first apply (1.2.1) with $h=2$, $a=q/t$, and $c=-q$, and then let $b,t\to 0$. Accordingly,

$$
\begin{aligned}
\sum_{n=0}^\infty \frac{(-1)^n q^{n^2}}{(q^4;q^4)_n(-q;q^2)_n} &= \sum_{n=0}^\infty \frac{(-1)^n q^{n^2}}{(q^2;q^2)_n(-q;q)_{2n}} \\
&= \frac{(q;q^2)_\infty}{(-q;q)_\infty} \sum_{n=0}^\infty \frac{q^{n(n+1)/2}}{(q;q)_n(q;q^2)_n} \\
&= \frac{1}{(-q;q)_\infty^2} \sum_{n=0}^\infty \frac{(-q;q)_n q^{n(n+1)/2}}{(q;q)_{2n}} =: S. \qquad (3.5.14)
\end{aligned}
$$

We now apply two representations in equation (81) of [262]. First, the right-hand side of (3.5.14) is equal to

$$
\begin{aligned}
S &= \frac{1}{(q^2;q^2)_\infty}\left((-q^{10},-q^{11},q^{21};q^{21})_\infty - q(-q^4,-q^{17},q^{21};q^{21})_\infty\right) \\
&= \frac{f(q^{10},q^{11}) - qf(q^4,q^{17})}{(q^2;q^2)_\infty},
\end{aligned}
$$

by the Jacobi triple product identity (1.4.8). Thus, we obtain the first equality in (3.5.13). The other representation of equation (81) in [262] yields from (3.5.14)

$$
S = \frac{1}{(q^2;q^2)_\infty}(q,q^6,q^7;q^7)_\infty (q^5,q^9;q^{14})_\infty
$$

$$
\begin{aligned}
&= \frac{1}{(q^2;q^2)_\infty} \frac{(q,q^6,q^7;q^7)_\infty (q^2,q^5;q^7)_\infty}{(q^2,q^{12};q^{14})_\infty} \\
&= \frac{(q^7;q^7)_\infty}{(q^2;q^2)_\infty} \frac{(q^2,q^5,q^7;q^7)_\infty}{(-q,-q^6,q^7;q^7)_\infty} \\
&= \frac{(q^7;q^7)_\infty f(-q^2,-q^5)}{(q^2;q^2)_\infty f(q,q^6)},
\end{aligned}
$$

upon two applications of the triple product identity (1.4.8). □

Entry 3.5.5 (p. 10). *With $f(a,b)$ defined by* (1.4.8),

$$
\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(q^4;q^4)_n(-q;q^2)_n} = \frac{f(q^8,q^{13}) - q^2 f(q,q^{20})}{(q^2;q^2)_\infty} = \frac{(q^7;q^7)_\infty f(-q^3,-q^4)}{(q^2;q^2)_\infty f(q^2,q^5)}. \tag{3.5.15}
$$

Proof. We apply (1.2.1) with $h=2$, $a=q^3/t$, and $c=-q$. We then let b and t tend to 0 to find that

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(q^4;q^4)_n(-q;q^2)_n} &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(q^2;q^2)_n(-q;q)_{2n}} \\
&= \frac{(q^3;q^2)_\infty}{(-q;q)_\infty} \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q;q)_n(q^3;q^2)_n} \\
&= \frac{1}{(-q;q)_\infty^2} \sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n+1)/2}}{(q;q)_{2n+1}} =: S. \qquad (3.5.16)
\end{aligned}
$$

We now apply the two representations for S on the far right side of (3.5.16) given in equation (80) of Slater's paper [262]. First,

$$
\begin{aligned}
S &= \frac{1}{(q^2;q^2)_\infty} \left((-q^8,-q^{13},q^{21};q^{21})_\infty - q^2(-q,-q^{20},q^{21};q^{21})_\infty \right) \\
&= \frac{f(q^8,q^{13}) - q^2 f(q,q^{20})}{(q^2;q^2)_\infty},
\end{aligned}
$$

by the triple product identity (1.4.8). This then proves the first equality in (3.5.15). Applying the second representation for S given in equation (80) of Slater's paper [262], we find that

$$
\begin{aligned}
S &= \frac{1}{(q^2;q^2)_\infty} (q^2,q^5,q^7;q^7)_\infty (q^3,q^{11};q^{14})_\infty \\
&= \frac{1}{(q^2;q^2)_\infty} \frac{(q^2,q^5,q^7;q^7)_\infty (q^3,q^4;q^7)_\infty}{(q^4,q^{10};q^{14})_\infty} \\
&= \frac{(q^7;q^7)_\infty}{(q^2;q^2)_\infty} \frac{(q^3,q^4,q^7;q^7)_\infty}{(-q^2,-q^5,q^7;q^7)_\infty}
\end{aligned}
$$

$$= \frac{(q^7;q^7)_\infty f(-q^3,-q^4)}{(q^2;q^2)_\infty f(q^2,q^5)},$$

upon two applications of (1.4.8). This then completes the proof of the second equality of (3.5.15). □

Entry 3.5.6 (p. 10). *If $f(a,b)$ is defined by* (1.4.8), *then*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(q^4;q^4)_n(-q;q^2)_{n+1}} = \frac{f(q^5,q^{16}) - qf(q^2,q^{19})}{(q^2;q^2)_\infty} = \frac{(q^7;q^7)_\infty f(-q,-q^6)}{(q^2;q^2)_\infty f(q^3,q^4)}. \tag{3.5.17}$$

Proof. Apply (1.2.1) with $h = 2$, $a = q^3/t$, and $c = -q^2$. Then let b and t tend to 0 and multiply both sides by $1/(1+q)$ to deduce that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(q^4;q^4)_n(-q;q^2)_{n+1}} &= \frac{1}{1+q}\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2+2n}}{(q^2;q^2)_n(-q^2;q)_{2n}} \\
&= \frac{(q^3;q^2)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty} \frac{q^{n(n+3)/2}}{(q;q)_n(q^3;q^2)_n} \\
&= \frac{1}{(-q;q)_\infty^2}\sum_{n=0}^{\infty} \frac{(-q;q)_n q^{n(n+3)/2}}{(q;q)_{2n+1}} =: S.
\end{aligned}$$

We now apply one of the equalities in equation (82) of Slater's paper [262] to S above and also apply the Jacobi triple product identity (1.4.8) to deduce that

$$\begin{aligned}
S &= \frac{1}{(q^2;q^2)_\infty}\left((-q^5,-q^{16},q^{21};q^{21})_\infty - q(-q^2,-q^{19},q^{21};q^{21})_\infty\right) \\
&= \frac{f(q^5,q^{16}) - qf(q^2,q^{19})}{(q^2;q^2)_\infty},
\end{aligned}$$

which yields the first equality in (3.5.17). We now invoke the second representation for S given in equation (82) in Slater's paper [262] to find that

$$\begin{aligned}
S &= \frac{1}{(q^2;q^2)_\infty}(q^3,q^4,q^7;q^7)_\infty(q,q^{13};q^{14})_\infty \\
&= \frac{1}{(q^2;q^2)_\infty}\frac{(q^3,q^4,q^7;q^7)_\infty(q,q^6;q^7)_\infty}{(q^6,q^8;q^{14})_\infty} \\
&= \frac{(q^7;q^7)_\infty}{(q^2;q^2)_\infty}\frac{(q,q^6,q^7;q^7)_\infty}{(-q^3,-q^4,q^7;q^7)_\infty} \\
&= \frac{(q^7;q^7)_\infty f(-q,-q^6)}{(q^2;q^2)_\infty f(q^3,q^4)},
\end{aligned}$$

upon two applications of the triple product identity (1.4.8). Thus, the second identity in (3.5.17) has been established. □

3.6 Miscellaneous Bilateral Identities

Entry 3.6.1 (p. 24).

$$(-q;-q)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(-q^2;q^2)_n} + \sum_{n=1}^{\infty} (-1;q^2)_n q^{n^2}. \tag{3.6.1}$$

Proof. In light of the fact that, by (3.1.1),

$$\frac{q^{2(-n)^2+(-n)}}{(-q^2;q^2)_{-n}} = (-1;q^2)_n q^{n^2},$$

we see that, with an application of (1.2.4),

$$\begin{aligned}
S :&= \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(-q^2;q^2)_n} + \sum_{n=1}^{\infty} (-1;q^2)_n q^{n^2} = \sum_{n=-\infty}^{\infty} \frac{q^{2n^2+n}}{(-q^2;q^2)_n} \\
&= \frac{1}{(-q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{2n^2+n} (-q^{2n+2};q^2)_\infty \\
&= \frac{1}{(-q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{2n^2+n} \sum_{m=0}^{\infty} \frac{q^{m^2+m+2nm}}{(q^2;q^2)_m} \\
&= \frac{1}{(-q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{2n^2+n} \left(\sum_{m=0}^{\infty} \frac{q^{4m^2+2m+4nm}}{(q^2;q^2)_{2m}} \right. \\
&\qquad \left. + \sum_{m=0}^{\infty} \frac{q^{4m^2+6m+2+4nm+2n}}{(q^2;q^2)_{2m+1}} \right).
\end{aligned}$$

Invert the order of summation. Then replace n by $n-m$ in the former sum and by $-n-1-m$ in the latter sum. Hence,

$$\begin{aligned}
S &= \frac{1}{(-q^2;q^2)_\infty} \left(\sum_{m=0}^{\infty} \frac{q^{2m^2+m}}{(q^2;q^2)_{2m}} \sum_{n=-\infty}^{\infty} q^{2n^2+n} \right. \\
&\qquad \left. + \sum_{m=0}^{\infty} \frac{q^{2m^2+3m+1}}{(q^2;q^2)_{2m+1}} \sum_{n=-\infty}^{\infty} q^{2n^2+n} \right) \\
&= \frac{1}{(-q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{2n^2+n} \left(\sum_{m=0}^{\infty} \frac{q^{2m(2m+1)/2}}{(q^2;q^2)_{2m}} + \sum_{m=0}^{\infty} \frac{q^{(2m+1)(2m+2)/2}}{(q^2;q^2)_{2m+1}} \right).
\end{aligned}$$

Now we apply (1.4.8) to deduce that

$$S = \frac{(-q;q^4)_\infty (-q^3;q^4)_\infty (q^4;q^4)_\infty}{(-q^2;q^2)_\infty} \sum_{m=0}^{\infty} \frac{q^{m(m+1)/2}}{(q^2;q^2)_m}$$

$$= (-q;q^2)_\infty (q^2;q^2)_\infty \sum_{m=0}^{\infty} \frac{q^{m(m+1)/2}}{(q^2;q^2)_m}$$
$$= (-q;-q)_\infty \sum_{m=0}^{\infty} \frac{q^{m(m+1)/2}}{(q^2;q^2)_m},$$

as desired. □

Entry 3.6.2 (p. 24).

$$(q^4;q^4)_\infty \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q^2;q^2)_n} = \sum_{n=0}^{\infty} (-q;q^2)_n q^{n^2+n} + \sum_{n=1}^{\infty} \frac{q^{2n^2-n}}{(-q;q^2)_n}. \tag{3.6.2}$$

Proof. First observe that, by (3.1.1),

$$\frac{q^{2(-n)^2-(-n)}}{(-q;q^2)_{-n}} = (-q;q^2)_n q^{n^2+n}.$$

We now proceed with the same steps that we used in the proof of Entry 3.6.1. We first use the equality above to write the right-hand side of (3.6.2) as a bilateral series. We next apply (1.2.4). Then invert the order of summation and replace n by $n-m$ in the former sum and by $-n-m$ in the latter sum. We again invert the order of summation and apply the Jacobi triple product identity (1.4.8). Accordingly, we find that

$$\sum_{n=0}^{\infty} q^{n^2+n}(-q;q^2)_n + \sum_{n=1}^{\infty} \frac{q^{2n^2-n}}{(-q;q^2)_n} = \sum_{n=-\infty}^{\infty} \frac{q^{2n^2-n}}{(-q;q^2)_n}$$
$$= \frac{1}{(-q;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{2n^2-n}(-q^{2n+1};q^2)_\infty$$
$$= \frac{1}{(-q;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{2n^2-n} \sum_{m=0}^{\infty} \frac{q^{m^2+2nm}}{(q^2;q^2)_m}$$
$$= \frac{1}{(-q;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{2n^2-n} \left(\sum_{m=0}^{\infty} \frac{q^{4m^2+4nm}}{(q^2;q^2)_{2m}} + \sum_{m=0}^{\infty} \frac{q^{4m^2+4m+1+4nm+2n}}{(q^2;q^2)_{2m+1}} \right)$$
$$= \frac{1}{(-q;q^2)_\infty} \left(\sum_{m=0}^{\infty} \frac{q^{2m^2+m}}{(q^2;q^2)_{2m}} \sum_{n=-\infty}^{\infty} q^{2n^2-n} \right.$$
$$\left. + \sum_{m=0}^{\infty} \frac{q^{2m^2+3m+1}}{(q^2;q^2)_{2m+1}} \sum_{n=-\infty}^{\infty} q^{2n^2-n} \right)$$
$$= \frac{1}{(-q;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{2n^2-n} \left(\sum_{m=0}^{\infty} \frac{q^{2m(2m+1)/2}}{(q^2;q^2)_{2m}} + \sum_{m=0}^{\infty} \frac{q^{(2m+1)(2m+2)/2}}{(q^2;q^2)_{2m+1}} \right)$$
$$= \frac{(-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty}{(-q;q^2)_\infty} \sum_{m=0}^{\infty} \frac{q^{m(m+1)/2}}{(q^2;q^2)_m}$$

$$= (q^4;q^4)_\infty \sum_{m=0}^{\infty} \frac{q^{m(m+1)/2}}{(q^2;q^2)_m},$$

as desired. □

We note that Entries 3.6.1 and 3.6.2 were first proved in [23].

Entry 3.6.3 (p. 27). *For* $a \neq 0$,

$$\sum_{n=-\infty}^{\infty} a^n q^{n^2/4} \sum_{m=0}^{\infty} \frac{(-1)^m a^m b^m q^{m^2/4}}{(q)_m} = (bq)_\infty$$
$$\times \left(a^{-1} q^{1/4} \sum_{n=0}^{\infty} \frac{a^{-2n} q^{n^2+n}}{(bq)_n} + a^{-1} q^{1/4} \sum_{n=1}^{\infty} (-1)^n a^{2n} b^n q^{n(n-1)/2} (1/b)_n \right)$$
$$+ (bq^{1/2})_\infty \left(\sum_{n=1}^{\infty} \frac{a^{-2n} q^{n^2}}{(bq^{1/2})_n} + \sum_{n=0}^{\infty} (-1)^n a^{2n} b^n q^{n^2/2} (q^{1/2}/b)_n \right). \tag{3.6.3}$$

Proof. In light of the facts that, by (3.1.1),

$$\frac{a^{-2(-n)} q^{(-n)^2+(-n)}}{(bq)_{-n}} = (-1)^n (1/b)_n a^{2n} b^n q^{n(n-1)/2}$$

and

$$\frac{a^{-2(-n)} q^{(-n)^2}}{(bq^{1/2})_{-n}} = (-1)^n (q^{1/2}/b)_n a^{2n} b^n q^{n^2/2},$$

we see that we may rewrite the right-hand side of (3.6.3) as

$$(bq)_\infty \sum_{n=-\infty}^{\infty} \frac{a^{-(2n+1)} q^{(2n+1)^2/4}}{(bq)_n} + (bq^{1/2})_\infty \sum_{n=-\infty}^{\infty} \frac{a^{-2n} q^{(2n)^2/4}}{(bq^{1/2})_n}$$
$$= \sum_{n=-\infty}^{\infty} a^{-n} q^{n^2/4} (bq^{(n+1)/2})_\infty =: S.$$

Applying (1.2.4), we find that

$$\begin{aligned} S &= \sum_{n=-\infty}^{\infty} a^{-n} q^{n^2/4} \sum_{m=0}^{\infty} \frac{(-1)^m b^m q^{(m^2+mn)/2}}{(q)_m} \\ &= \sum_{m=0}^{\infty} \frac{(-1)^m a^m b^m q^{m^2/4}}{(q)_m} \sum_{n=-\infty}^{\infty} a^{-(n+m)} q^{(n+m)^2/4} \\ &= \sum_{n=-\infty}^{\infty} a^n q^{n^2/4} \sum_{m=0}^{\infty} \frac{(-1)^m a^m b^m q^{m^2/4}}{(q)_m}, \end{aligned}$$

which is the left-hand side of (3.6.3). Hence the proof of Entry 3.6.3 is complete. □

Entry 3.6.4 (p. 4). *Recall that* $\varphi(q)$ *and* $\psi(q)$ *are defined by* (1.4.9) *and* (1.4.10), *respectively. Then*

$$\varphi(q)\left(2\sum_{n=-\infty}^{\infty}\frac{q^{n^2+n}}{1+q^{2n}}\right)-8\psi(q^2)\left(\sum_{n=1}^{\infty}\frac{q^{n^2}}{1+q^{2n-1}}\right)=\varphi^3(-q). \tag{3.6.4}$$

Proof. On the right-hand side below, we replace $m+n$ by m in the first inner sum, and by $-m$ in the second. Therefore,

$$\begin{aligned}S &:= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2/4+n/2}}{1+q^n}\sum_{m=-\infty}^{\infty}q^{m^2+mn+n^2/4}\\ &= \sum_{n=-\infty}^{\infty}\frac{q^{n^2+n}}{1+q^{2n}}\sum_{m=-\infty}^{\infty}q^{(m+n)^2}-\sum_{n=-\infty}^{\infty}\frac{q^{n^2}}{1+q^{2n-1}}\sum_{m=-\infty}^{\infty}q^{(n+m)^2-(m+n)}\\ &= \varphi(q)\sum_{n=-\infty}^{\infty}\frac{q^{n^2+n}}{1+q^{2n}}-4\psi(q^2)\sum_{n=1}^{\infty}\frac{q^{n^2}}{1+q^{2n-1}},\end{aligned}$$

where we have noted that, in the penultimate latter sum on n, the sum over $-\infty < n \leq 0$ is equal to that over $1 \leq n < \infty$. Thus, the left-hand side of (3.6.4) is equal to $2S$, and it remains to show that

$$2S = \varphi^3(-q). \tag{3.6.5}$$

Now, also

$$\begin{aligned}2S &= 2\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2/4+n/2}}{1+q^n}\sum_{m=-\infty}^{\infty}q^{m^2+mn+n^2/4}\\ &= 2\sum_{m=-\infty}^{\infty}(-1)^m q^{m(m-1)/2}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+m}q^{(n+m)(n+m+1)/2+m}}{q^m+q^{n+m}}\\ &= 2\sum_{m=-\infty}^{\infty}(-1)^m q^{m(m-1)/2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2+m}}{q^m+q^n}.\end{aligned} \tag{3.6.6}$$

By Entry 12.2.2 of Part I [31, p. 264] with $c=-q^{-m}$, we see that

$$\begin{aligned}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2+m}}{q^m+q^n} &= \frac{(q)_\infty^2}{(-q^{-m})_\infty(-q^{m+1})_\infty}\\ &= \frac{(q)_\infty^2}{2(-q)_\infty^2}q^{m(m+1)/2}.\end{aligned} \tag{3.6.7}$$

Putting (3.6.7) into (3.6.6) and using (1.4.9), we conclude that

$$\begin{aligned}2S &= 2\sum_{m=-\infty}^{\infty}(-1)^m q^{m(m-1)/2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2+m}}{q^m+q^n}\\ &= \frac{(q)_\infty^2}{(-q)_\infty^2}\sum_{m=-\infty}^{\infty}(-1)^m q^{m^2}\\ &= \varphi^3(-q).\end{aligned}$$

Hence, we have shown (3.6.5), and so the proof is complete. □

After stating Entry 3.6.4, Ramanujan remarks, "Generalisations simple and similar." If we carefully examine the previous proof, we see that the same method of proof yields the following theorem, which we state as an entry, because it most likely is what Ramanujan had in mind.

Entry 3.6.5 (p. 4). *If k is any positive integer, then*

$$\begin{aligned}&\sum_{j=0}^{2k-1}(-1)^j\left(\sum_{m=-\infty}^{\infty}q^{km^2+jm}\right)\sum_{n=-\infty}^{\infty}\frac{q^{(2kn+j+1)(2kn+j)/2-kn^2-jn}}{1+q^{2kn+j}}\\ &\quad= \varphi^2(-q)\varphi(-q^k).\end{aligned} \tag{3.6.8}$$

The identity (3.6.8) reduces to Entry 3.6.4 when $k=1$.

We close this chapter with an entry that has close ties to the results in Chapter 12 of [31]. However, a nontrivial application of (3.2.2) suggests that we place it here.

Entry 3.6.6 (p. 29). *Recall that $\varphi(q)$ and $\psi(q)$ are defined in* (1.4.9) *and* (1.4.10), *respectively. Then*

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(-q^4;q^4)_n} = \frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n q^{n^2}}{(-q^2;q^2)_n^2} + \frac{1}{2}\frac{\varphi^2(q)}{\psi(q)}. \tag{3.6.9}$$

Proof. We replace q by $-q$ and a by i in Entry 12.3.4 of [31, p. 267] to deduce that

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(-q^4;q^4)_n} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-q)^{n(n+1)/2}}{1+q^{2n}} + \frac{(-q;q^2)_\infty\varphi(q)}{2(-q^2;q^2)_\infty}. \tag{3.6.10}$$

By the product representations given in (1.4.9) and (1.4.10) and Euler's identity,

$$\begin{aligned}\frac{(-q;q^2)_\infty\varphi(q)}{(-q^2;q^2)_\infty} &= \frac{(-q;q^2)_\infty\varphi^2(q)}{(-q^2;q^2)_\infty(-q;q^2)_\infty^2(q^2;q^2)_\infty}\\ &= \frac{(q;q^2)_\infty\varphi^2(q)}{(q^2;q^2)_\infty}\\ &= \frac{\varphi^2(q)}{\psi(q)}.\end{aligned} \tag{3.6.11}$$

Using (3.6.11) in (3.6.10) and comparing the result with (3.6.9), we see that in order to complete our proof we must show that

$$\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_nq^{n^2}}{(-q^2;q^2)_n^2}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-q)^{n(n+1)/2}}{1+q^{2n}}. \tag{3.6.12}$$

Now by Entry 12.2.1 of [31, p. 264] with $c=1$,

$$\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+n}}{1+q^{2n}}=\frac{1}{2}\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_nq^{n^2}}{(-q^2;q^2)_n^2}. \tag{3.6.13}$$

Comparing (3.6.13) with (3.6.12), we see that it now suffices to show that

$$\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+n}}{1+q^{2n}}=\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-q)^{n(n+1)/2}}{1+q^{2n}}. \tag{3.6.14}$$

Replacing q by $-q$ in (3.6.14), we now see that we must show that

$$\begin{aligned}(-q;q^2)_\infty\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{2n^2+n}}{1+q^{2n}}&=(q;q^2)_\infty\sum_{n=-\infty}^{\infty}\frac{q^{n(n+1)/2}}{1+q^{2n}}\\&=(q;q^2)_\infty\mathrm{Re}\sum_{n=-\infty}^{\infty}\frac{q^{n(n+1)/2}}{1+iq^n},\end{aligned} \tag{3.6.15}$$

where it is assumed that q is real.

We now consider (3.2.2) with $\alpha=-q/\tau$, $\beta=-i$, $z=\gamma=\tau$, and $\delta=-iq$. Letting $\tau\to0$, we find that

$$(1+i)\sum_{n=-\infty}^{\infty}\frac{q^{n(n+1)/2}}{1+iq^n}=2\frac{(-q;q)_\infty^2}{(-q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{i^nq^{n(n+1)/2}}{1+q^n}.$$

Thus,

$$\begin{aligned}\mathrm{Re}\sum_{n=-\infty}^{\infty}\frac{q^{n(n+1)/2}}{1+iq^n}&=\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\mathrm{Re}\left((1-i)\sum_{n=-\infty}^{\infty}\frac{i^nq^{n(n+1)/2}}{1+q^n}\right)\\&=\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\left(\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(2n+1)}}{1+q^{2n}}+\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{(2n+1)(n+1)}}{1+q^{2n+1}}\right)\\&=\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(2n+1)}}{1+q^{2n}},\end{aligned}$$

since the second sum in the penultimate line vanishes, because replacing the index n by $-n-1$ reveals that it is equal to its negative. Hence, we see that we have proved (3.6.15), which therefore completes the proof. ∎

4

Well-Poised Series

4.1 Introduction

Among Ramanujan's most far-reaching and striking discoveries are the Rogers–Ramanujan identities, given for $|q| < 1$ by [241], [31, Chapter 10]

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \tag{4.1.1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{4.1.2}$$

In the lost notebook, we find many identities of the Rogers–Ramanujan type; see, for example, Chapter 11 of our first book [31]. The vast majority of them can be proved as special limiting cases of Watson's q-analogue of Whipple's theorem [274], [151, p. 242, equation (III.18)]. If α, β, γ, δ, and ϵ are any complex numbers such that $\beta\gamma\delta\epsilon \neq 0$, and if N is any nonnegative integer, then

$$\begin{aligned} &{}_8\phi_7 \left(\begin{matrix} \alpha, \ q\sqrt{\alpha}, \ -q\sqrt{\alpha}, \ \beta, \ \gamma, \ \delta, \ \epsilon, \ q^{-N} \\ \sqrt{\alpha}, \ -\sqrt{\alpha}, \ \dfrac{\alpha q}{\beta}, \dfrac{\alpha q}{\gamma}, \dfrac{\alpha q}{\delta}, \dfrac{\alpha q}{\epsilon}, \alpha q^{N+1} \end{matrix} ; q, \frac{\alpha^2 q^{N+2}}{\beta\gamma\delta\epsilon} \right) \\ &\qquad = \frac{(\alpha q)_N \left(\dfrac{\alpha q}{\delta\epsilon}\right)_N}{\left(\dfrac{\alpha q}{\delta}\right)_N \left(\dfrac{\alpha q}{\epsilon}\right)_N} \, {}_4\phi_3 \left(\begin{matrix} \dfrac{\alpha q}{\beta\gamma}, \delta, \epsilon, q^{-N} \\ \dfrac{\alpha q}{\beta}, \dfrac{\alpha q}{\gamma}, \dfrac{\delta\epsilon q^{-N}}{\alpha} \end{matrix} ; q, q \right). \end{aligned} \tag{4.1.3}$$

Although we cannot find a statement of this theorem in Ramanujan's works, he recorded many deductions from it. In particular, see [54, p. 16, Entry 7] and the pages immediately following.

The series ${}_8\phi_7$ in (4.1.3) is called very well-poised. The term "well poised" refers to the fact that the product of each column of entries is the same, in

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this case αq. In the first column, α is to be paired with q, arising from $(q;q)_n$, which is in the denominator of ${}_8\phi_7$ but which does not appear in the notation ${}_8\phi_7$. The adverb "very" refers to the fact that the second and third columns are

$$\begin{array}{c} q\sqrt{\alpha},\ -q\sqrt{\alpha} \\ \sqrt{\alpha},\ \ -\sqrt{\alpha}. \end{array}$$

Two lesser known, but highly useful, identities of this nature were proved by W.N. Bailey. They are [41, equation (6.3)]

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(\rho_1;q)_n(\rho_2;q)_n(aq/f;q^2)_n}{(q;q)_n(aq;q^2)_n(aq/f;q)_n}\left(\frac{aq}{\rho_1\rho_2}\right)^n = \frac{(aq/\rho_1;q)_\infty(aq/\rho_2;q)_\infty}{(aq;q)_\infty(aq/(\rho_1\rho_2);q)_\infty}\Bigg(1 \\ &+\sum_{n=1}^{\infty} \frac{(aq^2;q^2)_{n-1}(f;q^2)_n(\rho_1;q)_{2n}(\rho_2;q)_{2n}(1-aq^{4n})}{(q^2;q^2)_n(aq^2/f;q^2)_n(aq/\rho_1;q)_{2n}(aq/\rho_2;q)_{2n}}\left(\frac{a^3}{\rho_1^2\rho_2^2 f}\right)^n q^{2n^2+2n}\Bigg) \end{aligned} \tag{4.1.4}$$

and [41, equation (6.1)]

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(\rho_1;q^2)_n(\rho_2;q^2)_n(-aq/b;q)_{2n}}{(q^2;q^2)_n(a^2q^2/b^2;q^2)_n(-aq;q)_{2n}}\left(\frac{a^2q^2}{\rho_1\rho_2}\right)^n \\ &= \frac{(a^2q^2/\rho_1;q^2)_\infty(a^2q^2/\rho_2;q^2)_\infty}{(a^2q^2;q^2)_\infty(a^2q^2/(\rho_1\rho_2);q^2)_\infty} \\ &\times\left(1+\sum_{n=1}^{\infty} \frac{(aq;q)_{n-1}(b;q)_n(\rho_1;q^2)_n(\rho_2;q^2)_n(1-aq^{2n})}{(q;q)_n(aq/b;q)_n(a^2q^2/\rho_1;q^2)_n(a^2q^2/\rho_2;q^2)_n}\left(\frac{a^3q^2}{\rho_1\rho_2 b}\right)^n q^{n^2}\right). \end{aligned} \tag{4.1.5}$$

We conclude this introduction by stating two limiting formulas that we use many times in the sequel, usually without comment:

$$\lim_{N\to\infty} \frac{(q^{-N})_n}{(\alpha q^{-N})_n} = \frac{1}{\alpha^n},$$

$$\lim_{N\to\infty} \frac{(q^{-N})_n q^{Nn}}{(\alpha q^{N+1})_n} = (-1)^n q^{n(n-1)/2}.$$

4.2 Applications of (4.1.3)

Ramanujan recorded many formulas that are direct corollaries of (4.1.3). We begin this section with the formulas that he published in [241]. Entry 4.2.3 is the most general special case of (4.1.3) that appears in the lost notebook. Indeed, many of the subsequent entries are instances of Entry 4.2.3. We have, however, chosen for coherence and consistency to deduce each of the 14 entries in this section directly from (4.1.3).

Entry 4.2.1 (p. 41). *For any complex number a,*

$$(aq)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n (a)_n (1-aq^{2n})}{(q)_n (1-a)} a^{2n} q^{n(5n-1)/2}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n (aq)_n (1-a^2 q^{4n+2})}{(q)_n} a^{2n} q^{n(5n+1)/2}. \qquad (4.2.1)$$

These are the identities from which Ramanujan deduced (4.1.1) and (4.1.2) in his joint paper with L.J. Rogers [241].

Proof. The first line of this entry follows directly from (4.1.3) by letting $\alpha = a$ and letting β, γ, δ, ϵ, and N tend to ∞. This is precisely the argument used by G.N. Watson [274].

We now divide both identities in (4.2.1) by $(aq)_\infty$, and so we need to prove that $R_1(a,q) = R_2(a,q)$, where

$$R_1(a,q) := \sum_{n=0}^{\infty} \frac{(-1)^n (1-aq^{2n})}{(q)_n (aq^n)_\infty} a^{2n} q^{n(5n-1)/2}$$

and

$$R_2(a,q) := \sum_{n=0}^{\infty} \frac{(-1)^n (1-a^2q^{4n+2})}{(q)_n (aq^{n+1})_\infty} a^{2n} q^{n(5n+1)/2}.$$

We follow Ramanujan's lead from [241]. Thus,

$$\begin{aligned}
R_1(a,q) &= \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(5n-1)/2} \{(1-q^n) + q^n(1-aq^n)\}}{(q)_n (aq^n)_\infty} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n a^{2n} q^{n(5n-1)/2}}{(q)_{n-1} (aq^n)_\infty} + \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(5n+1)/2}}{(q)_n (aq^{n+1})_\infty} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} a^{2n+2} q^{n(5n+1)/2+4n+2}}{(q)_n (aq^{n+1})_\infty} + \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(5n+1)/2}}{(q)_n (aq^{n+1})_\infty} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(5n+1)/2} (1-a^2q^{4n+2})}{(q)_n (aq^{n+1})_\infty} \\
&= R_2(a,q),
\end{aligned}$$

which is what we wanted to show. $\square$

Entry 4.2.2 (p. 41). *For any complex number a,*

$$(aq)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n^2+n}}{(q)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n (aq)_n (1-aq^{2n+1})}{(q)_n} a^{2n} q^{n(5n+3)/2}.$$

Proof. Replace a by aq in the first line of (4.2.1) and then multiply both sides by $(1-aq)$. $\square$

Entry 4.2.3 (p. 42). *For arbitrary complex numbers a, b, and c,*

$$\begin{aligned}\frac{(aq)_\infty}{(-bq)_\infty}\sum_{n=0}^{\infty}&\frac{(-aq/b)_n b^n q^{n(n+1)/2}}{(q)_n(-cq)_n}\\&=\sum_{n=0}^{\infty}\frac{(-1)^n(aq)_n(-aq/b)_n(-aq/c)_n(1-aq^{2n+1})b^nc^nq^{n(3n+1)/2}}{(q)_n(-bq)_n(-cq)_n}.\end{aligned}$$

Proof. In (4.1.3), set $\alpha=aq$, $\gamma=-aq/c$, and $\delta=-aq/b$. Then let β, ϵ, and N tend to ∞. After multiplying both sides by $(1-aq)$, the result simplifies to the desired identity. $\square$

Entry 4.2.4 (p. 28). *If a and b are arbitrary, then*

$$\begin{aligned}\frac{(-aq)_\infty}{(-bq)_\infty}\sum_{n=0}^{\infty}&\frac{(aq/b)_n b^n q^{n(n+1)/2}}{(q)_n}\\&=\sum_{n=0}^{\infty}\frac{(aq/b)_n(-aq)_n(1+aq^{2n+1})a^nb^nq^{2n^2+n}}{(q)_n(-bq)_n}.\end{aligned}$$

Proof. In (4.1.3), let $\alpha=-aq$ and $\delta=aq/b$, and then let β, γ, ϵ, and $N\to\infty$. Multiply both sides of the resulting equality by $(1+aq)$ and simplify to complete the proof. $\square$

It may be noted that the result above also follows from Entry 4.2.3 by replacing a by $-a$ and setting $c=0$.

Entry 4.2.5 (p. 28). *For any complex numbers a and λ,*

$$(\lambda q)_\infty\sum_{n=0}^{\infty}\frac{\lambda^n q^{n^2}}{(q)_n(aq)_n}=1+\sum_{n=0}^{\infty}\frac{(\lambda/a)_n(\lambda q)_{n-1}(1-\lambda q^{2n})\lambda^na^nq^{2n^2}}{(q)_n(aq)_n}.$$

Proof. In (4.1.3), replace α by λ, then set $\beta=\lambda/a$, and finally let γ, δ, ϵ, and N tend to ∞. Upon algebraic simplification, the desired result follows. $\square$

The following entry was recorded (with one misprint) by I. Pak [226, Equation (4.4.2)] in his survey paper. See also Entry 1.7.10.

Entry 4.2.6 (p. 41). *Recall that $\psi(q)$ is defined by* (1.4.10). *Then*

$$\frac{1}{\psi(q)}=\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_nq^{n^2}}{(q^2;q^2)_n^2}.$$

Proof. In (4.1.3), replace q by q^2, then set $\delta = q$, next let β, ϵ, and N tend to ∞, and lastly let α and γ tend to 1. After simplification, we arrive at

$$1 = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(q^2;q^2)_n^2}. \tag{4.2.2}$$

Multiplying both sides by $(q;q^2)_\infty/(q^2;q^2)_\infty$ and invoking the product representation from (1.4.10), we complete the proof. □

We might note that in fact, one may deduce this result from (1.2.9), wherein one replaces q by q^2, then sets $a = q$, $b = q/t$, and $c = q^2$, and finally lets $t \to 0$.

Entry 4.2.7 (p. 41). *If* $\varphi(q)$ *and* $\psi(q)$ *are defined by* (1.4.9) *and* (1.4.10), *respectively, then*

$$\frac{\varphi(q^3)}{\psi(q)} = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(q^4;q^4)_n}.$$

This identity is given by L.J. Slater [262] as equation (25) with q replaced by $-q$.

Proof. In (4.1.3), replace q by q^2, let $\alpha \to 1$, then set $\beta = -1$ and $\delta = q$, and let γ, ϵ, and N tend to ∞. The result after simplification is

$$1 + 2\sum_{n=1}^{\infty} q^{3n^2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(q^2;q^2)_n (-q^2;q^2)_n},$$

and this reduces to the desired result upon invoking (1.4.9) and (1.4.10). □

The next entry is a corrected version of one of Slater's identities [262, equation (6)].

Entry 4.2.8 (p. 34). *We have*

$$\frac{\varphi(-q^3)}{\varphi(-q)} = \sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n^2}}{(q;q)_n (q;q^2)_n}.$$

Proof. In (4.1.3), set $\beta = \sqrt{q}$ and $\gamma = -\sqrt{q}$, let δ, ϵ, and $N \to \infty$, and let $\alpha \to 1$. Hence, using the definition of $f(a,b)$ and the Jacobi triple product identity in (1.4.8), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1;q)_n q^{n^2}}{(q;q)_n (q;q^2)_n} &= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} (1+q^n) q^{n(3n-1)/2} \\
&= \frac{f(q,q^2)}{(q;q)_\infty} \\
&= \frac{(-q;q^3)_\infty (-q^2;q^3)_\infty (q^3;q^3)_\infty}{(q;q)_\infty} \\
&= \frac{(q^3;q^3)_\infty}{(-q^3;q^3)_\infty} \frac{(-q;q)_\infty}{(q;q)_\infty} = \frac{\varphi(-q^3)}{\varphi(-q)},
\end{aligned}$$

by (1.4.9). □

Entry 4.2.9 (p. 34). *We have*

$$\frac{\varphi(-q^3)}{\varphi(-q)}=\sum_{n=0}^{\infty}\frac{(-q;q)_nq^{n^2}}{(q;q)_n(q;q^2)_{n+1}}.$$

Proof. For each integer $r\geq 0$, we define

$$f_r(q):=\sum_{n=0}^{\infty}\frac{(-1;q)_nq^{(n+r)^2}}{(q;q)_n(q;q^2)_{n+r}}-\sum_{n=0}^{\infty}\frac{(-q;q)_nq^{(n+r)^2}}{(q;q)_n(q;q^2)_{n+r+1}}. \tag{4.2.3}$$

Hence,

$$\begin{aligned}f_r(q)&=\sum_{n=0}^{\infty}\frac{(-q;q)_{n-1}q^{(n+r)^2}}{(q;q)_n(q;q^2)_{n+r+1}}\left(2(1-q^{2n+2r+1})-(1+q^n)\right)\\&=\sum_{n=1}^{\infty}\frac{(-q;q)_{n-1}q^{(n+r)^2}}{(q;q)_n(q;q^2)_{n+r+1}}\left((1-q^n)-2q^{2n+2r+1}\right)\\&=\sum_{n=0}^{\infty}\frac{(-q;q)_nq^{(n+r+1)^2}}{(q;q)_n(q;q^2)_{n+r+2}}-\sum_{n=0}^{\infty}\frac{(-1;q)_nq^{(n+r+1)^2}}{(q;q)_n(q;q^2)_{n+r+1}}\\&=-f_{r+1}(q).\end{aligned}$$

But clearly $\lim_{r\to\infty}f_r(q)=0$. Therefore, the recurrence formula above implies that $f_0(q)=0$. Hence, by (4.2.3) with $r=0$ and Entry 4.2.8,

$$\sum_{n=0}^{\infty}\frac{(-q;q)_nq^{n^2}}{(q;q)_n(q;q^2)_{n+1}}=\sum_{n=0}^{\infty}\frac{(-1;q)_nq^{n^2}}{(q;q)_n(q;q^2)_n}-f_0(q)=\frac{\varphi(-q^3)}{\varphi(-q)}.$$

□

The following entry is due to Slater [262, equation (48)].

Entry 4.2.10 (p. 34). *We have*

$$\frac{\varphi(q^3)}{\varphi(-q^2)}=\sum_{n=0}^{\infty}\frac{(-1;q^2)_nq^{n(n+1)}}{(q;q)_{2n}}.$$

Proof. In (4.1.3), replace q by q^2, then set $\beta=q$ and $\delta=-1$, let γ, ϵ, and $N\to\infty$, and lastly let $\alpha\to 1$. After simplification, we find that

$$\sum_{n=0}^{\infty}\frac{(-1;q^2)_nq^{n(n+1)}}{(q;q)_{2n}}=\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\left(1+2\sum_{n=1}^{\infty}q^{3n^2}\right)=\frac{\varphi(q^3)}{\varphi(-q^2)},$$

by (1.4.9). □

Entry 4.2.11 was rediscovered by D. Stanton [269].

Entry 4.2.11 (p. 41). *If* $f(a,b)$ *is defined by* (1.4.8) *and* $\psi(q)$ *is defined by* (1.4.10), *then*

$$\frac{f(q,q^5)}{\psi(q)}=\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_nq^{n(n+2)}}{(q^4;q^4)_n}.$$

Proof. In (4.1.3), replace q by q^2, then set $\alpha=q^2$, $\beta=-q^2$, and $\delta=q$, and lastly let γ, ϵ, and N approach ∞. After multiplying both sides by $1+q=(1-q^2)/(1-q)$ and simplifying, we find that

$$\sum_{n=0}^{\infty}(1+q^{2n+1})q^{3n^2+2n}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_nq^{n(n+2)}}{(q^2;q^2)_n(-q^2;q^2)_n},$$

and this reduces to the desired result by invoking (1.4.10) and noting that

$$q^{3(-n-1)^2+2(-n-1)}=q^{3n^2+4n+1}.$$

∎

Entry 4.2.12 can be found in Slater's compendium [262, equation (22)] as well as Bailey's book [44, p. 72, equation (10)].

Entry 4.2.12 (p. 34). *We have*

$$\frac{f(-q,-q^5)}{\varphi(-q)}=\sum_{n=0}^{\infty}\frac{(-q;q)_nq^{n^2+n}}{(q;q)_n(q;q^2)_{n+1}}.$$

Proof. In (4.1.3), set $\alpha=q$, $\beta=\sqrt{q}$, and $\gamma=-\sqrt{q}$, and then let δ, ϵ, and N tend to ∞. After multiplying both sides by $1/(1-q)$, simplifying, and eventually using (1.4.10), (1.4.8), and (1.4.9) in turn, we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-q;q)_nq^{n^2+n}}{(q;q)_n(q;q^2)_{n+1}}&=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}q^{3n(n+1)/2}\\&=\frac{(q^6;q^6)_\infty}{(q;q)_\infty(q^3;q^6)_\infty}\\&=\frac{(-q;q)_\infty}{(q;q)_\infty}\frac{(q^6;q^6)_\infty}{(-q;q)_\infty(q^3;q^6)_\infty}\\&=\frac{(-q;q)_\infty}{(q;q)_\infty}\frac{(q^6;q^6)_\infty(q;q^2)_\infty}{(q^3;q^6)_\infty}\\&=\frac{(-q;q)_\infty}{(q;q)_\infty}(q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty\\&=\frac{f(-q,-q^5)}{\varphi(-q)}.\end{aligned}$$

∎

One can find the next entry in Slater's paper [262, equation (28)].

Entry 4.2.13 (p. 35). *We have*

$$\frac{f(q,q^5)}{\varphi(-q^2)} = \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n(n+1)}}{(q;q)_{2n+1}}.$$

Proof. In (4.1.3), we replace q by q^2, then we set $\alpha = q^2$, $\beta = q$, and $\delta = -q^2$, and let γ, ϵ, and N tend to ∞. Multiplying both sides by $(1-q^2)/(1-q) = (1+q)$ and simplifying, we find that

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n(n+1)}}{(q;q)_{2n+1}} = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} q^{3n^2+2n}(1+q^{2n+1}) = \frac{f(q,q^5)}{\varphi(-q^2)},$$

where we used the same calculation that we used in the proof of Entry 4.2.11, as well as (1.4.9). □

Entry 4.2.14 (p. 41). *We have*

$$\frac{1}{\psi(q)} \sum_{n=0}^{\infty} (-1)^n q^{3n^2+2n}(1+q^{2n+1}) = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n(n+2)}}{(q^2;q^2)_n^2}.$$

Proof. In (4.1.3), replace q by q^2, then set $\alpha = \gamma = q^2$ and $\delta = q$, and finally let β, ϵ, and $N \to \infty$. After multiplying both sides by $(1+q) = (1-q^2)/(1-q)$ and simplifying, we find that

$$\sum_{n=0}^{\infty} (-1)^n q^{3n^2+2n}(1+q^{2n+1}) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n(n+2)}}{(q^2;q^2)_n^2},$$

and the desired result follows from invoking (1.4.10). □

If we replace q by $-q$ in the next entry, we obtain the analytic version of the first Göllnitz–Gordon identity [157], [159]. Observe that Entry 4.2.15 is identical to Entry 1.7.11, for which we gave a different proof.

Entry 4.2.15 (p. 41). *We have*

$$\frac{\psi(q^4)}{f(q,q^7)} = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(q^2;q^2)_n}.$$

Proof. In (4.1.3), replace q by q^2, then let $\alpha \to 1$, set $\delta = q$, and let β, γ, ϵ, and N tend to ∞. After simplification, we find that

$$\sum_{n=0}^{\infty} q^{4n^2-n}(1+q^{2n}) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(q^2;q^2)_n}.$$

Next we use the Jacobi triple product identity (1.4.8) to observe that

$$\begin{aligned}
\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty} q^{4n^2-n}(1+q^{2n}) &= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} f(q^3,q^5) \\
&= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}(-q^3;q^8)_\infty(-q^5;q^8)_\infty(q^8;q^8)_\infty \\
&= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\frac{(-q;q^2)_\infty(q^8;q^8)_\infty}{(-q;q^8)_\infty(-q^7;q^8)_\infty} \\
&= \frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty}\frac{(q^2;q^4)_\infty}{(-q^4;q^4)_\infty(q^2;q^2)_\infty(-q;q^8)_\infty(-q^7;q^8)_\infty} \\
&= \frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty}\frac{1}{(-q;q^8)_\infty(-q^7;q^8)_\infty(q^8;q^8)_\infty} \\
&= \frac{\psi(q^4)}{f(q,q^7)},
\end{aligned}$$

by (1.4.10) and (1.4.8). □

Entry 4.2.16 (p. 202). *Let a and b be any complex numbers. Then*

$$\begin{aligned}
(abq)_\infty \sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2}}{(-aq)_n(-bq)_n} \\
= (1+a)(1+b)\sum_{n=0}^{\infty} \frac{(-1)^n (abq)_{n-1}(1-abq^{2n})a^n b^n q^{n(3n+1)/2}}{(q)_n(1+aq^n)(1+bq^n)}.
\end{aligned}$$

Proof. In (4.1.3), set $\alpha = ab$, $\beta = -a$, and $\gamma = -b$. Then let δ, ϵ, and N tend to ∞. The desired result then follows upon simplification. □

4.3 Applications of Bailey's Formulas

Entry 4.3.1 (p. 26). *Let a and b be any complex numbers. Then*

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-aq/b;q)_n b^n q^{n(n+1)/2}}{(q;q)_n(aq^2;q^2)_n} \\
= \frac{(-bq;q)_\infty}{(aq;q)_\infty}\sum_{n=0}^{\infty} \frac{(-1)^n (aq;q^2)_n(-aq/b;q)_{2n}(1-aq^{4n+1})a^n b^{2n} q^{5n^2+n}}{(q^2;q^2)_n(-bq;q)_{2n}}.
\end{aligned}$$

Proof. In (4.1.4), replace a by aq, set $\rho_2 = -aq/b$, and let f and ρ_1 tend to ∞. □

Entry 4.3.2 (p. 41). *If a is any complex number, then*

$$\sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(-aq^2;q^2)_n} = \sum_{n=0}^{\infty} \frac{(q;q^2)_n(1+aq^{4n+1})a^{3n}q^{5n^2+n}}{(-aq^2;q^2)_n}.$$

Proof. Replace a by $-a$ in Entry 4.3.1 and then set $b=a$. □

Entry 4.3.3 (p. 28). *For arbitrary complex numbers a and b,*

$$\sum_{n=0}^{\infty} \frac{(-bq;q^2)_n a^n q^{n(n+1)/2}}{(-aq^2;q^2)_n(-bq;q)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n(aq/b;q^2)_n(q;q^2)_n(1+aq^{4n+1})a^{2n}b^nq^{4n^2+n}}{(-aq^2;q^2)_n(-bq^2;q^2)_n}.$$

Proof. In (4.1.4), replace a by $-aq$, then set $\rho_2=q$ and $f=aq/b$, and lastly let $\rho_1\to\infty$. □

Entry 4.3.4 (p. 3). *For $|aq|<1$,*

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n(aq;q^2)_n a^n q^n}{(-aq;q)_{2n+1}} = \sum_{n=0}^{\infty}(-1)^n a^n q^{n(n+1)}.$$

Proof. In (4.1.5), set $\rho_1=a$, $\rho_2=q^2$, and $b=-a$. After simplification, we arrive at

$$\sum_{n=0}^{\infty} \frac{(a;q^2)_n(q;q^2)_n a^n}{(-aq;q)_{2n}} = (1+a)\left(1+\sum_{n=1}^{\infty}(-1)^n a^n q^{n^2}\right),$$

and this becomes the desired result if we replace a by aq and then divide both sides by $(1+aq)$. □

Entry 4.3.5 (p. 28). *If a and b are arbitrary complex numbers, then*

$$\sum_{n=0}^{\infty} \frac{(-1)^n(a^2q^2/b;q^2)_n b^n q^{n(n+1)}}{(q^2;q^2)_n(-aq;q)_{2n+1}} = \frac{(bq^2;q^2)_\infty}{(a^2q^2;q^2)_\infty}\sum_{n=0}^{\infty} \frac{(aq;q)_n(a^2q^2/b;q^2)_n(1-aq^{2n+1})a^nb^nq^{n(5n+3)/2}}{(q;q)_n(bq^2;q^2)_n}.$$

Proof. In (4.1.5), let b and ρ_2 tend to ∞. Replace a by aq and then set $\rho_1=a^2q^2/b$. Finally, multiply both sides by $1/(1+aq)$ to complete the proof. □

Entry 4.3.6 (p. 28). *If a is arbitrary, then*

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(n+1)}}{(-aq)_{2n+1}} = \sum_{n=0}^{\infty} \frac{(-q)_n(1-aq^{2n+1})a^{3n}q^{n(5n+3)/2}}{(-aq)_n}.$$

Proof. Set $b = a^2$ in the previous entry and simplify. □

The next entry as listed by Ramanujan does not contain enough terms to completely determine each side. However, the following interpretation is consistent with what Ramanujan has written.

Entry 4.3.7 (p. 26). *Let a and b be arbitrary complex numbers. Then*

$$\sum_{n=0}^{\infty} \frac{(-1)^n (a^2q^2/b;q^2)_n b^n q^{n^2+n}}{(q^2;q^2)_n(-aq;q)_{2n}} \\ = \frac{(bq^2;q^2)_\infty}{(a^2q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(aq;q)_n(a^2q^2/b;q^2)_n(1-a^2q^{4n+2})a^n b^n q^{n(5n+1)/2}}{(q;q)_n(bq^2;q^2)_n}. \tag{4.3.1}$$

Proof. In (4.3.1), replace b by a^2q^2/b and multiply both sides by $(-aq;q)_\infty$. Then (4.3.1) is equivalent to the assertion

$$L_2(a) = R_2(a),$$

where

$$L_2(a) := \sum_{n=0}^{\infty} \frac{(-1)^n (b;q^2)_n (-aq^{2n+1};q)_\infty a^{2n} b^{-n} q^{n^2+3n}}{(q^2;q^2)_n}$$

and

$$R_2(a) := \frac{(a^2q^4/b;q^2)_\infty}{(aq;q)_\infty} \sum_{n=0}^{\infty} \frac{(aq;q)_n(b;q^2)_n(1-a^2q^{4n+2})a^{3n}b^{-n}q^{5n(n+1)/2}}{(q;q)_n(a^2q^4/b;q^2)_n}.$$

We now rewrite Entry 4.3.5 by replacing b by a^2q^2/b therein and multiplying both sides by $(-aq;q)_\infty$. The revision of Entry 4.3.5 now takes the form

$$L_1(a) = R_1(a),$$

where

$$L_1(a) := \sum_{n=0}^{\infty} \frac{(-1)^n (b;q^2)_n (-aq^{2n+2};q)_\infty a^{2n} b^{-n} q^{n(n+3)}}{(q^2;q^2)_n}$$

and

$$R_1(a) := \frac{(a^2q^4/b;q^2)_\infty}{(aq;q)_\infty} \sum_{n=0}^{\infty} \frac{(aq;q)_n(b;q^2)_n(1-aq^{2n+1})a^{3n}b^{-n}q^{n(5n+7)/2}}{(q;q)_n(a^2q^4/b;q^2)_n}.$$

Our object now is to prove that $L_2(a) = R_2(a)$ in light of the fact that $L_1(a) = R_1(a)$.

First,

$$L_2(a) - aqL_2(aq) = \sum_{n=0}^{\infty} \frac{(-1)^n (b;q^2)_n (-aq^{2n+2};q)_\infty a^{2n} b^{-n} q^{n(n+3)}}{(q^2;q^2)_n} \times \left(1 + aq^{2n+1} - aq^{2n+1}\right) = L_1(a). \tag{4.3.2}$$

Secondly,

$$\begin{aligned}
&R_2(a) - R_1(a) \\
&= \sum_{n=0}^{\infty} \frac{(b;q^2)_n (a^2q^{2n+4}/b;q^2)_\infty a^{3n} b^{-n} q^{5n(n+1)/2}}{(q;q)_n (aq^{n+1};q)_\infty} \\
&\quad \times \left\{(1 - a^2q^{4n+2}) - q^n(1 - aq^{2n+1})\right\} \\
&= \sum_{n=0}^{\infty} \frac{(b;q^2)_n (a^2q^{2n+4}/b;q^2)_\infty a^{3n} b^{-n} q^{5n(n+1)/2}}{(q;q)_n (aq^{n+1};q)_\infty} \\
&\quad \times \left\{(1 - q^n) + aq^{3n+1}(1 - aq^{n+1})\right\} \\
&= \sum_{n=1}^{\infty} \frac{(b;q^2)_n (a^2q^{2n+4}/b;q^2)_\infty a^{3n} b^{-n} q^{5n(n+1)/2}}{(q;q)_{n-1} (aq^{n+1};q)_\infty} \\
&\quad + \sum_{n=0}^{\infty} \frac{(b;q^2)_n (a^2q^{2n+4}/b;q^2)_\infty a^{3n+1} b^{-n} q^{5n(n+1)/2+3n+1}}{(q;q)_n (aq^{n+2};q)_\infty} \\
&= \sum_{n=0}^{\infty} \frac{(b;q^2)_{n+1} (a^2q^{2n+6}/b;q^2)_\infty a^{3n+3} b^{-n-1} q^{5n(n+1)/2+5n+5}}{(q;q)_n (aq^{n+2};q)_\infty} \\
&\quad + \sum_{n=0}^{\infty} \frac{(b;q^2)_n (a^2q^{2n+4}/b;q^2)_\infty a^{3n+1} b^{-n} q^{5n(n+1)/2+3n+1}}{(q;q)_n (aq^{n+2};q)_\infty} \\
&= aq \sum_{n=0}^{\infty} \frac{(b;q^2)_n (a^2q^{2n+6}/b;q^2)_\infty (aq)^{3n} b^{-n} q^{5n(n+1)/2}}{(q;q)_n (aq^{n+2};q)_\infty} \\
&\quad \times \left\{a^2b^{-1}q^{2n+4}(1 - bq^{2n}) + (1 - a^2q^{2n+4}/b)\right\} \\
&= aq \sum_{n=0}^{\infty} \frac{(b;q^2)_n ((aq)^2q^{2n+4}/b;q^2)_\infty (1 - (aq)^2q^{4n+2}) (aq)^{3n} b^{-n} q^{5n(n+1)/2}}{(q;q)_n ((aq)q^{n+1};q)_\infty} \\
&= aqR_2(aq).
\end{aligned} \tag{4.3.3}$$

Now, from (4.3.2),

$$L_2(a) = L_1(a) + aq\, L_2(aq),$$

and by iteration we may deduce that

$$L_2(a) = \sum_{j=0}^{\infty} (aq)^j L_1(aq^j).$$

We then proved in (4.3.3) that

$$R_2(a) = R_1(a) + aq\, R_2(aq),$$

and again by iteration we may conclude that

$$R_2(a) = \sum_{j=0}^{\infty} (aq)^j R_1(aq^j).$$

But we know that $R_1(a) = L_1(a)$ from Entry 4.3.5. Therefore,

$$L_2(a) = \sum_{j=0}^{\infty} (aq)^j L_1(aq^j) = \sum_{j=0}^{\infty} (aq)^j R_1(aq^j) = R_2(a),$$

and this is the desired result. $\square$

Entry 4.3.8 (p. 28). *If a is any complex number, then*

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(n+1)}}{(-aq)_{2n}} = \sum_{n=0}^{\infty} \frac{(-q)_n (1 - a^2 q^{4n+2}) a^{3n} q^{n(5n+1)/2}}{(-aq)_n}. \tag{4.3.4}$$

Proof. Let us call the left- and right-hand sides of (4.3.4) $L_8(a)$ and $R_8(a)$, respectively. Thus, we need to prove that

$$L_8(a) = R_8(a). \tag{4.3.5}$$

Also, let $L_6(a)$ and $R_6(a)$ denote the left- and right-hand sides of Entry 4.3.6. In particular,

$$L_6(a) := \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(n+1)}}{(-aq)_{2n+1}}.$$

Then, we see that

$$\frac{a^2q^2}{1+aq} L_6(aq) = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+2} q^{n^2+3n+2}}{(-aq)_{2n+2}} = 1 - L_8(a). \tag{4.3.6}$$

On the other hand, since

$$R_6(a) := \sum_{n=0}^{\infty} \frac{(-q)_n (1 - aq^{2n+1}) a^{3n} q^{n(5n+3)/2}}{(-aq)_n},$$

we find that

$$R_8(a) + \frac{a^2q^2}{1+aq} R_6(aq)$$
$$= \sum_{n=0}^{\infty} \frac{(-q)_n (1 - a^2 q^{4n+2}) a^{3n} q^{n(5n+1)/2}}{(-aq)_n}$$

$$+\sum_{n=0}^{\infty}\frac{(-q)_n(1-aq^{2n+2})a^{3n+2}q^{n(5n+9)/2+2}}{(-aq)_{n+1}}$$
$$=\sum_{n=0}^{\infty}\frac{(-q)_n a^{3n}q^{n(5n+1)/2}}{(-aq)_n}-\sum_{n=0}^{\infty}\frac{(-q)_n a^{3n+2}q^{n(5n+9)/2+2}}{(-aq)_n}$$
$$+\sum_{n=0}^{\infty}\frac{(-q)_n a^{3n+2}q^{n(5n+9)/2+2}}{(-aq)_{n+1}}-\sum_{n=1}^{\infty}\frac{(-q)_{n-1}a^{3n}q^{n(5n+3)/2}}{(-aq)_n}.$$

Now combine the first and fourth sums and the second and third sums on the right-hand side above to deduce that

$$\begin{aligned}
R_8(a)&+\frac{a^2q^2}{1+aq}R_6(aq)\\
&=1+\sum_{n=1}^{\infty}\frac{(-q)_{n-1}a^{3n}q^{n(5n+1)/2}}{(-aq)_n}\left((1+q^n)-q^n\right)\\
&\quad-\sum_{n=0}^{\infty}\frac{(-q)_n a^{3n+2}q^{n(5n+9)/2+2}}{(-aq)_{n+1}}\left((1+aq^{n+1})-1\right)\\
&=1+\sum_{n=1}^{\infty}\frac{(-q)_{n-1}a^{3n}q^{n(5n+1)/2}}{(-aq)_n}-\sum_{n=1}^{\infty}\frac{(-q)_{n-1}a^{3n}q^{n(5n+1)/2}}{(-aq)_n}\\
&=1.
\end{aligned}$$

Hence, using (4.3.6), Entry 4.3.6, and the last equality above, we conclude that

$$\begin{aligned}
L_8(a)&=1-\frac{a^2q^2}{1+aq}L_6(aq)\\
&=1-\frac{a^2q^2}{1+aq}R_6(aq)\\
&=1-1+R_8(a)=R_8(a).
\end{aligned}$$

Thus, (4.3.5) has been demonstrated, as desired. □

Entry 4.3.9 (p. 27). *If a is an arbitrary complex number, then*

$$(aq;q)_\infty\sum_{n=0}^{\infty}\frac{a^nq^{n^2}}{(q;q)_n(aq;q^2)_n}=\sum_{n=0}^{\infty}\frac{(-1)^n(aq;q^2)_n(1-a^2q^{8n+2})a^{3n}q^{7n^2}}{(q^2;q^2)_n}.$$

The instance $a=1$ of Entry 4.3.9 is identity (61) and the instance $a=q^2$ is identity (59) in Slater's paper [262]. These identities also appear in Rogers's paper [248], but the case in which $a=1$ first appeared in Rogers's earlier paper [247]. Our proof is based on the work of A. Sills [261].

Proof. We define, for $i = 1, 2, 3$,

$$S_i(a;q) := \sum_{n=0}^{\infty} \frac{a^n q^{n^2+(3-i)n}}{(q;q)_n (aq;q^2)_{n+[(5-i)/3]}}.$$

First, by inspection,

$$S_1(a;q) = \frac{S_3(aq^2;q)}{1-aq}. \tag{4.3.7}$$

Next,

$$\begin{aligned} S_2(a;q) - S_1(a;q) &= \sum_{n=1}^{\infty} \frac{(1-q^n)a^n q^{n^2+n}}{(q;q)_n (aq;q^2)_{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{a^{n+1} q^{n^2+3n+2}}{(q;q)_n (aq;q^2)_{n+2}} = \frac{aq^2}{1-aq} S_2(aq^2;q), \end{aligned} \tag{4.3.8}$$

and

$$\begin{aligned} S_3(a;q) - S_2(a;q) &= \sum_{n=0}^{\infty} \frac{(1-aq^{2n+1}-q^n)a^n q^{n^2}}{(q;q)_n (aq;q^2)_{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{a^n q^{n^2}}{(q;q)_{n-1} (aq;q^2)_{n+1}} - aq \sum_{n=0}^{\infty} \frac{a^n q^{n^2+2n}}{(q;q)_n (aq;q^2)_{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{a^{n+2} q^{(n+2)^2}}{(q;q)_n (aq;q^2)_{n+2}} \\ &= \frac{a^2 q^4}{1-aq} S_1(aq^2;q). \end{aligned} \tag{4.3.9}$$

Iteration of the three functional equations (4.3.7)–(4.3.9) together with the initial values $S_i(0;q) = 1$ reveals that these three equations uniquely define $S_i(0;q)$, $i = 1, 2, 3$.

Next, we note that if [8]

$$Q_{k,i}(a;q) := \sum_{n=0}^{\infty} \frac{(-1)^n (1 - a^i q^{(2n+1)i}) a^{kn} q^{(2k+1)n(n+1)/2 - in}}{(q;q)_n (aq^{n+1})_\infty}, \tag{4.3.10}$$

then

$$Q_{k,i}(a;q) - Q_{k,i-1}(a;q) = a^{i-1} q^{i-1} Q_{k,k-i+1}(aq;q). \tag{4.3.11}$$

Now define, for $i = 1, 2, 3$,

$$T_i(a;q) := \frac{Q_{3,i}(a;q^2)}{(aq;q^2)_\infty},$$

from which it immediately follows that

$$T_i(a;0) = 1, \qquad 1 \le i \le 3.$$

Using (4.3.11) with $k = 3$ and $i = 1$ and the fact that $Q_{3,0}(a;q) \equiv 0$, which is immediately deducible from the definition (4.3.10), we find that

$$T_1(a;q) = \frac{T_3(aq^2;q)}{1-aq}. \tag{4.3.12}$$

Using (4.3.11) with q replaced by q^2 and with $k = 3$ and $i = 2$, we find that

$$T_2(a;q) - T_1(a;q) = \frac{aq^2}{1-aq} T_2(aq^2;q). \tag{4.3.13}$$

Lastly, using (4.3.11) with $k = i = 3$ and q replaced by q^2, we deduce that

$$T_3(a;q) - T_2(a;q) = \frac{a^2q^4}{1-aq} T_1(aq^2;q). \tag{4.3.14}$$

Thus, $T_i(a;q)$, $i = 1,2,3$, satisfies the same initial conditions and functional equations (4.3.12)–(4.3.14) as those satisfied by $S_i(a;q)$ in (4.3.7)–(4.3.9). Hence, for $i = 1,2,3$,

$$T_i(a;q) = S_i(a;q).$$

What is important for us is this assertion for $i = 2$, because the assertion of Entry 4.3.9 is equivalent to

$$(aq;q)_\infty T_2(a/q;q) = (aq;q)_\infty S_2(a/q;q).$$

Hence, the proof is complete. $\square$

5

Bailey's Lemma and Theta Expansions

5.1 Introduction

Most of the entries to be established in this chapter were originally proved in [22]. That paper appeared before the discoveries presented in [24] were made. It is now possible to present these results in a way that makes clear their relationship to the hierarchy of q-hypergeometric identities growing out of Bailey's lemma [41, equation (3.1)].

In Section 5.2, we prove the two central lemmas of [22] by means of Bailey's lemma. These are identities (5.2.3) and (5.2.4). Section 5.3 is devoted to the corollaries of (5.2.3), and Section 5.4 to those of (5.2.4).

The term "theta expansions" refers to the fact that (5.2.3) and (5.2.4) each involve partial products related to Jacobi's triple product identity (1.4.8).

5.2 The Main Lemma

A pair of sequences $\{\alpha_n\}$ and $\{\beta_n\}$ related by the identity

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}} \tag{5.2.1}$$

is called a *Bailey pair*. For a Bailey pair $\{\alpha_n\}$ and $\{\beta_n\}$, Bailey's lemma [41, equation (3.1)] is given by

$$\begin{aligned}\sum_{n=0}^{\infty} (\rho_1)_n(\rho_2)_n \left(\frac{aq}{\rho_1\rho_2}\right)^n \beta_n \\ = \frac{(aq/\rho_1)_\infty (aq/\rho_2)_\infty}{(aq)_\infty (aq/(\rho_1\rho_2))_\infty} \sum_{n=0}^{\infty} \frac{(\rho_1)_n(\rho_2)_n}{(aq/\rho_1)_n(aq/\rho_2)_n} \left(\frac{aq}{\rho_1\rho_2}\right)^n \alpha_n.\end{aligned} \tag{5.2.2}$$

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Bailey's proof consists in substituting the expression for β_n from (5.2.1) into the left-hand side of (5.2.2), interchanging the order of summation, and then summing the interior series by invoking (1.2.9) with $t = c/(ab)$.

We now construct two Bailey pairs. First, with $a = 1$ and q replaced by q^2, we set $\alpha_0 = 1$ and

$$\alpha_N = (x^N + x^{-N})q^{N^2}, \qquad N \geq 1.$$

Then, using (1.7.3) in the penultimate step below, we find that

$$\begin{aligned}
\beta_N &= \frac{1}{(q^2;q^2)_N^2} + \sum_{r=1}^{N} \frac{(x^r + x^{-r})q^{r^2}}{(q^2;q^2)_{N-r}(q^2;q^2)_{N+r}} \\
&= \sum_{r=-N}^{N} \frac{x^r q^{r^2}}{(q^2;q^2)_{N-r}(q^2;q^2)_{N+r}} \\
&= \frac{1}{(q^2;q^2)_{2N}} \sum_{r=-N}^{N} \begin{bmatrix} 2N \\ N+r \end{bmatrix}_{q^2} x^r q^{r^2} \\
&= \frac{1}{(q^2;q^2)_{2N}} \sum_{r=0}^{2N} \begin{bmatrix} 2N \\ r \end{bmatrix}_{q^2} x^{r-N} q^{(r-N)^2} \\
&= \frac{x^{-N}q^{N^2}}{(q^2;q^2)_{2N}} (-xq^{-2N+1};q^2)_{2N} \\
&= \frac{(-xq;q^2)_N(-x^{-1}q;q^2)_N}{(q^2;q^2)_{2N}}.
\end{aligned}$$

Inserting this pair into (5.2.2) and recalling that q has been replaced by q^2 and that $a = 1$, we obtain Lemma 1 from [22], which is given by

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(\rho_1;q^2)_n(\rho_2;q^2)_n(-xq;q^2)_n(-x^{-1}q;q^2)_n}{(q^2;q^2)_{2n}} \left(\frac{q^2}{\rho_1\rho_2}\right)^n \qquad (5.2.3) \\
&= \frac{(q^2/\rho_1;q^2)_\infty(q^2/\rho_2;q^2)_\infty}{(q^2;q^2)_\infty(q^2/(\rho_1\rho_2);q^2)_\infty} \\
&\quad \times \left(1 + \sum_{N=1}^{\infty} \frac{(\rho_1;q^2)_N(\rho_2;q^2)_N(x^N + x^{-N})\rho_1^{-N}\rho_2^{-N}q^{N^2+2N}}{(q^2/\rho_1;q^2)_N(q^2/\rho_2;q^2)_N}\right) \\
&= \frac{(q^2/\rho_1;q^2)_\infty(q^2/\rho_2;q^2)_\infty}{(q^2;q^2)_\infty(q^2/(\rho_1\rho_2);q^2)_\infty} \sum_{N=-\infty}^{\infty} \frac{(\rho_1;q^2)_N(\rho_2;q^2)_N}{(q^2/\rho_1;q^2)_N(q^2/\rho_2;q^2)_N} \left(\frac{xq^2}{\rho_1\rho_2}\right)^N q^{N^2}.
\end{aligned}$$

Next we construct a Bailey pair now with $a = q$ and

$$\alpha_N = (x^{N+1} + x^{-N})q^{N(N+1)/2}, \qquad N \geq 0.$$

Hence, using (1.7.3) in the penultimate step below, we deduce that

$$\begin{aligned}
\beta_N &= (1-q)\sum_{r=0}^{N} \frac{(x^{r+1}+x^{-r})q^{r(r+1)/2}}{(q)_{N-r}(q)_{N+r+1}} \\
&= \frac{(1-q)}{(q)_{2N+1}} \sum_{r=0}^{N} \begin{bmatrix} 2N+1 \\ N-r \end{bmatrix} (x^{r+1}+x^{-r})q^{r(r+1)/2} \\
&= \frac{1}{(q^2)_{2N}} \sum_{r=-N-1}^{N} \begin{bmatrix} 2N+1 \\ N-r \end{bmatrix} x^{-r}q^{r(r+1)/2} \\
&= \frac{1}{(q^2)_{2N}} \sum_{r=0}^{2N+1} \begin{bmatrix} 2N+1 \\ r \end{bmatrix} x^{-N+r}q^{(N-r)(N-r+1)/2} \\
&= \frac{x^{-N}q^{N(N+1)/2}}{(q^2)_{2N}} \sum_{r=0}^{2N+1} \begin{bmatrix} 2N+1 \\ r \end{bmatrix} x^{r}q^{r(r-1)/2-rN} \\
&= \frac{x^{-N}q^{N(N+1)/2}}{(q^2)_{2N}} (-xq^{-N})_{2N+1} \\
&= \frac{(-x)_{N+1}(-q/x)_N}{(q^2)_{2N}}.
\end{aligned}$$

Inserting this pair into (5.2.2), recalling that $a = q$, and dividing both sides by $(1-q)$, we obtain Lemma 2 from [22] given by

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-x)_{n+1}(-q/x)_n(\rho_1)_n(\rho_2)_n}{(q)_{2n+1}} \left(\frac{q^2}{\rho_1\rho_2}\right)^n \qquad (5.2.4)\\
&= \frac{(q^2/\rho_1)_\infty(q^2/\rho_2)_\infty}{(q)_\infty(q^2/(\rho_1\rho_2))_\infty} \sum_{n=0}^{\infty} \frac{(\rho_1)_n(\rho_2)_n}{(q^2/\rho_1)_n(q^2/\rho_2)_n}(x^{n+1}+x^{-n})\left(\frac{q^2}{\rho_1\rho_2}\right)^n q^{n(n+1)/2}.
\end{aligned}$$

5.3 Corollaries of (5.2.3)

Entry 5.3.1 (p. 33). *Recall that $f(a,b)$ is defined in* (1.4.8). *Then, for $a \neq 0$,*

$$\sum_{n=0}^{\infty} \frac{(-aq;q^2)_n(-q/a;q^2)_n}{(q^2;q^2)_{2n}} q^{2n^2} = \frac{f(aq^3,q^3/a)}{(q^2;q^2)_\infty}. \qquad (5.3.1)$$

First Proof of Entry 5.3.1. In (5.2.3), set $x = a$ and let ρ_1 and ρ_2 tend to ∞. ∎

In her thesis [225], Padmavathamma gave a different proof of Entry 5.3.1, independent of Bailey's theorems. We need two preliminary results: one is a decomposition found in P.A. MacMahon's book [216, p. 75], and the other a corollary of the q-Gauss summation theorem, Entry 1.3.1.

Lemma 5.3.1. *If i and j are arbitrary nonnegative integers, then*

$$(-aq;q^2)_i(-q/a;q^2)_j = \frac{(q^2;q^2)_{i+j}}{(q^2;q^2)_i(q^2;q^2)_j} + \sum_{m=1}^{i} \frac{(q^2;q^2)_{i+j}a^m q^{m^2}}{(q^2;q^2)_{i-m}(q^2;q^2)_{j+m}} + \sum_{m=1}^{j} \frac{(q^2;q^2)_{i+j}a^{-m} q^{m^2}}{(q^2;q^2)_{i+m}(q^2;q^2)_{j-m}}.$$

Lemma 5.3.2. *If $|a| < 1$,*

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2-n}}{(a)_n(q)_n} = \frac{1}{(a)_\infty}.$$

First Proof of Entry 5.3.1. Letting b and c tend to 0 in (1.3.8), we immediately deduce the desired result. □

If we replace a by aq in the last lemma, we obtain a classical generating function for the partition function $p(n)$ due to Euler.

Second Proof of Entry 5.3.1. Multiplying both sides of (5.3.1) by $(q^2;q^2)_\infty$, we write the identity to be proved in the equivalent form

$$F(a;q) := \sum_{n=0}^{\infty}(-aq;q^2)_n(-q/a;q^2)_n(q^{4n+2};q^2)_\infty q^{2n^2} = f(aq^3, q^3/a). \tag{5.3.2}$$

Invoking Lemma 5.3.1 with $i = j = n$, we find that

$$(-aq;q^2)_n(-q/a;q^2)_n = \frac{(q^2;q^2)_{2n}}{(q^2;q^2)_n^2} + \sum_{m=1}^{n} c(m,n)(a^m + a^{-m}), \tag{5.3.3}$$

where

$$c(m,n) := \frac{(q^2;q^2)_{2n} q^{m^2}}{(q^2;q^2)_{n+m}(q^2;q^2)_{n-m}}. \tag{5.3.4}$$

Note that

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2} = \sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty},$$

where $p(n)$ denotes the ordinary partition function, and so

$$(q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n^2} = 1. \tag{5.3.5}$$

Putting (5.3.3) in (5.3.2), inverting the order of summation, employing (5.3.5), recalling (5.3.4), replacing n by $m + k$, and employing Lemma 5.3.2 with q replaced by q^2 and $a = q^{4m+2}$, we find that

$$
\begin{aligned}
F(a;q) &= \sum_{n=0}^{\infty} \frac{(q^2;q^2)_{2n}(q^{4n+2};q^2)_\infty}{(q^2;q^2)_n^2} q^{2n^2} \\
&\quad + \sum_{n=0}^{\infty} (q^{4n+2};q^2)_\infty q^{2n^2} \sum_{m=1}^{n} c(m,n)(a^m + a^{-m}) \\
&= (q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_n^2} \\
&\quad + \sum_{m=1}^{\infty} (a^m + a^{-m}) \sum_{n=m}^{\infty} c(m,n)(q^{4n+2};q^2)_\infty q^{2n^2} \\
&= 1 + \sum_{m=1}^{\infty} (a^m + a^{-m}) \sum_{n=m}^{\infty} \frac{(q^{4n+2};q^2)_\infty (q^2;q^2)_{2n}}{(q^2;q^2)_{m+n}(q^2;q^2)_{n-m}} q^{2n^2+m^2} \\
&= 1 + (q^2;q^2)_\infty \sum_{m=1}^{\infty} (a^m + a^{-m}) q^{m^2} \sum_{n=m}^{\infty} \frac{q^{2n^2}}{(q^2;q^2)_{n+m}(q^2;q^2)_{n-m}} \\
&= 1 + (q^2;q^2)_\infty \sum_{m=1}^{\infty} (a^m + a^{-m}) q^{3m^2} \sum_{k=0}^{\infty} \frac{q^{2k^2+4mk}}{(q^2;q^2)_{2m+k}(q^2;q^2)_k} \\
&= 1 + (q^2;q^2)_\infty \sum_{m=1}^{\infty} \frac{(a^m + a^{-m}) q^{3m^2}}{(q^2;q^2)_{2m}} \sum_{k=0}^{\infty} \frac{q^{(4m+2)k} q^{2k(k-1)}}{(q^{4m+2};q^2)_k (q^2;q^2)_k} \\
&= 1 + (q^2;q^2)_\infty \sum_{m=1}^{\infty} \frac{(a^m + a^{-m}) q^{3m^2}}{(q^2;q^2)_{2m}(q^{4m+2};q^2)_\infty} \\
&= 1 + \sum_{m=1}^{\infty} (a^m + a^{-m}) q^{3m^2} \\
&= f(aq^3, q^3/a).
\end{aligned}
$$

Thus, (5.3.2) has been established, and so the proof is complete. □

The next entry is equation (29) on L.J. Slater's list [262].

Entry 5.3.2 (p. 33). *If* $f(a,b)$ *and* $\psi(q)$ *are defined by* (1.4.8) *and* (1.4.10), *respectively, then*

$$
\sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q;q)_{2n}} q^{n^2} = \frac{f(q^2,q^4)}{\psi(-q)}.
$$

Proof. First, observe that

$$
\sum_{n=-\infty}^{\infty} q^{3(2n+1)^2/2} i^{2n+1} = 0. \tag{5.3.6}
$$

Second, in Entry 5.3.1, set $a = i$ and then replace q by $\sqrt{q}$. In the resulting sum on the right-hand side, we may replace n by $2n$, because the terms with odd index sum to 0 by (5.3.6). Consequently, eventually using (1.4.9) and (1.4.8), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q;q)_{2n}} q^{n^2} &= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} q^{6n^2} i^{2n} \\
&= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{6n^2} \\
&= \frac{(q^6;q^6)_\infty}{(-q^6;q^6)_\infty (q;q)_\infty} \\
&= \frac{(q^6;q^6)_\infty (-q^2;q^6)_\infty (-q^4;q^6)_\infty (-q^6;q^6)_\infty}{(-q^6;q^6)_\infty (-q^2;q^2)_\infty (q;q^2)_\infty (q^2;q^2)_\infty} \\
&= \frac{f(q^2,q^4)}{(q;q^4)_\infty (q^3;q^4)_\infty (q^4;q^4)_\infty} \\
&= \frac{f(q^2,q^4)}{f(-q,-q^3)} = \frac{f(q^2,q^4)}{\psi(-q)},
\end{aligned}$$

by (1.4.10). ∎

Entry 5.3.3 (p. 33). *If* $f(a,b)$ *and* $\psi(q)$ *are defined by* (1.4.8) *and* (1.4.10), *respectively, then*

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n^2}{(q^2;q^2)_{2n}} q^{2n^2} = \frac{f(q,q^2)}{\psi(q)}.$$

Proof. Set $a = -1$ in Entry 5.3.1. Using (1.4.9), Euler's identity, (1.4.10), and (1.4.8), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(q;q^2)_n^2}{(q^2;q^2)_{2n}} q^{2n^2} &= \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2} \\
&= \frac{(q^3;q^3)_\infty}{(-q^3;q^3)_\infty (q^2;q^2)_\infty} \\
&= \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \frac{(q^3;q^3)_\infty (-q;q)_\infty}{(-q^3;q^3)_\infty} \\
&= \frac{(-q;q^3)_\infty (-q^2;q^3)_\infty (q^3;q^3)_\infty}{\psi(q)} \\
&= \frac{f(q,q^2)}{\psi(q)}.
\end{aligned}$$

∎

The next entry is kindred to two identities of Slater [262, equations (124) and (125)] but does not appear in her list.

Entry 5.3.4 (p. 33, corrected). *If* $f(a,b)$ *and* $f(-q)$ *are defined by* (1.4.8) *and* (3.1.3), *respectively, then*

$$\sum_{n=0}^{\infty} \frac{(q^3;q^6)_n}{(q;q^2)_n(q^2;q^2)_{2n}} q^{2n^2} = \frac{f(-q,-q^5)}{f(-q)}(q^9;q^{18})_\infty. \tag{5.3.7}$$

Ramanujan's function $\chi(q)$ is defined by

$$\chi(q) := (-q;q^2)_\infty.$$

Thus, the last factor on the right-hand side of (5.3.7) can be written as $\chi(-q^9)$. There does not seem to be any advantage to expressing (5.3.7), or any other identity in the early chapters of this volume, in terms of $\chi(q)$, however.

Proof. First note that

$$(e^{2\pi i/3}q;q^2)_n(e^{-2\pi i/3}q;q^2)_n = \frac{(q^3;q^6)_n}{(q;q^2)_n}, \tag{5.3.8}$$

and that, by (1.4.8),

$$\begin{aligned} f(-e^{2\pi i/3}q^3, -e^{-2\pi i/3}q^3) &= (e^{2\pi i/3}q^3;q^6)_\infty(e^{-2\pi i/3}q^3;q^6)_\infty(q^6;q^6)_\infty \\ &= \frac{(q^9;q^{18})_\infty(q^6;q^6)_\infty}{(q^3;q^6)_\infty}. \end{aligned} \tag{5.3.9}$$

In Entry 5.3.1, set $a = -e^{2\pi i/3}$. Hence, using (5.3.8), (5.3.9), and (1.4.8), we find that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(q^3;q^6)_n}{(q;q^2)_n(q^2;q^2)_{2n}} q^{2n^2} &= \frac{1}{(q^2;q^2)_\infty} f(-e^{2\pi i/3}q^3, -e^{-2\pi i/3}q^3) \\ &= \frac{(q^9;q^{18})_\infty(q^6;q^6)_\infty}{(q^3;q^6)_\infty(q^2;q^2)_\infty} \\ &= \frac{(q;q^6)_\infty(q^3;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty(q^9;q^{18})_\infty}{(q^3;q^6)_\infty(q;q^2)_\infty(q^2;q^2)_\infty} \\ &= \frac{f(-q,-q^5)}{f(-q)}(q^9;q^{18})_\infty. \end{aligned}$$

□

The following entry can be obtained from a result in Andrews's paper [15, p. 526, equation (1.9)] by replacing q by q^2, setting $b = -aq$, and setting $c = q$.

Entry 5.3.5 (p. 33). *If* $f(a,b)$ *and* $\psi(q)$ *are defined by* (1.4.8) *and* (1.4.10), *respectively, and if* $a \neq 0$, *then*

$$\sum_{n=0}^{\infty} \frac{(-aq;q^2)_n(-a^{-1}q;q^2)_n}{(q;q^2)_n(q^4;q^4)_n} q^{n^2} = \frac{f(aq^2,q^2/a)}{\psi(-q)}.$$

Proof. In (5.2.3), set $x = a$ and $\rho_1 = -q$, and let $\rho_2 \to \infty$. Consequently, after simplification,

$$\sum_{n=0}^{\infty} \frac{(-aq;q^2)_n(-a^{-1}q;q^2)_n}{(q;q^2)_n(q^4;q^4)_n} q^{n^2} = \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{N=-\infty}^{\infty} a^N q^{2N^2} = \frac{f(aq^2,q^2/a)}{\psi(-q)},$$

by (1.4.10) and (1.4.8). □

Different proofs of Entries 5.3.1–5.3.5 were devised by Padmavathamma [225]. The next entry can be found in Slater's paper [262, equation (4)].

Entry 5.3.6 (p. 33). *If* $f(-q)$ *is defined by* (3.1.3), *then*

$$\sum_{n=0}^{\infty} \frac{(\pm 1)^n(-q;q^2)_n}{(q^4;q^4)_n} q^{n^2} = \frac{f(\pm q, \pm q^2)}{f(-q^4)}.$$

Proof. First we consider the case in which we take the minus sign. Here replace q by $-q$ and then set $a = -1$ in Entry 5.3.5. Thus, using (1.4.10), the Jacobi triple product identity (1.4.8), and (1.4.9), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n(-q;q^2)_n}{(q^4;q^4)_n} q^{n^2} &= \frac{f(-q^2,-q^2)}{\psi(q)} = \frac{\varphi(-q^2)}{f(q,q^3)} \\
&= \frac{(q^2;q^2)_\infty}{(-q^2;q^2)_\infty(-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty} \\
&= \frac{(q^2;q^2)_\infty}{(-q;q)_\infty(q^4;q^4)_\infty} = \frac{(q;q)_\infty}{(q^4;q^4)_\infty} = \frac{f(-q)}{f(-q^4)},
\end{aligned}$$

by (3.1.3). Since $f(-q) = f(-q,-q^2)$, the proof of the first case is finished.

The case for the plus sign follows from a special case of Entry 4.2.3. Namely, divide both sides of Entry 4.2.3 by $(1-aq)$, then replace q by q^2, and lastly set $a = 1/q^2$, $b = 1/q$, and $c = 1$. Multiplying both sides by $(-q;q^2)_\infty/(q^2;q^2)_\infty$, simplifying considerably, and using (1.4.9) and (1.4.8), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^4;q^4)_n} q^{n^2} &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \left(1 + 2\sum_{n=1}^{\infty} (-1)^n q^{3n^2}\right) \\
&= \frac{(-q;q^2)_\infty(q^3;q^3)_\infty}{(q^2;q^2)_\infty(-q^3;q^3)_\infty} \\
&= \frac{(-q;q)_\infty(q^3;q^3)_\infty}{(q^4;q^4)_\infty(-q^3;q^3)_\infty} \\
&= \frac{(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty}{(q^4;q^4)_\infty} \\
&= \frac{f(q,q^2)}{f(-q^4)},
\end{aligned}$$

which is what we wanted to prove. □

Entry 5.3.7 (p. 33). *If* $\psi(q)$ *is defined by* (1.4.10) *and* $f(-q)$ *is defined by* (3.1.3), *then*

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^4;q^4)_n} q^{n^2+2n} = \frac{\psi(q^3)}{f(-q^4)}.$$

We include this entry in this chapter because of its similarity to Entry 5.3.6, even though its proof is based on an identity in Chapter 4. Observe that in Entry 4.2.11, recorded eight pages earlier in his lost notebook, Ramanujan wrote Entry 5.3.7 in a slightly different form.

Proof. In Entry 4.2.3, replace q by q^2 and set $a = 1$, $b = q$, and $c = 1$. Multiply both sides by $1/(1+q)$. Then multiply both sides by $(-q;q^2)_\infty/(q^2;q^2)_\infty$. After simplification and using (1.4.8), we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-q;q^2)_n}{(q^4;q^4)_n} q^{n^2+2n} &= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} (-1)^n (1-q^{2n+1}) q^{3n^2+2n} \\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} f(-q,-q^5) \\
&= \frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} (q;q^6)_\infty (q^5;q^6)_\infty (q^6;q^6)_\infty \\
&= \frac{(-q;q^2)_\infty (q;q^2)_\infty (q^6;q^6)_\infty}{(q^2;q^2)_\infty (q^3;q^6)_\infty} \\
&= \frac{(q^6;q^6)_\infty}{(q^4;q^4)_\infty (q^3;q^6)_\infty} \\
&= \frac{\psi(q^3)}{f(-q^4)},
\end{aligned}$$

by (1.4.10). □

Entry 5.3.8 (p. 33). *We have*

$$\sum_{n=0}^{\infty} \frac{(-q^3;q^6)_n}{(q^2;q^2)_{2n}} q^{n^2} = \frac{\psi(-q^2)}{\psi(-q)} (-q^6;q^{12})_\infty.$$

Proof. In Entry 5.3.5, set $a = e^{2\pi i/3}$. Observe that

$$\frac{(-e^{2\pi i/3}q;q^2)_n(-e^{-2\pi i/3}q;q^2)_n}{(q;q^2)_n(q^4;q^4)_n} = \frac{(-q^3;q^6)_n}{(-q;q^2)_n(q;q^2)_n(q^4;q^4)_n} = \frac{(-q^3;q^6)_n}{(q^2;q^2)_{2n}},$$

and, by (1.4.8),

$$\begin{aligned}
f(e^{2\pi i/3}q^2, e^{-2\pi i/3}q^2) &= (-e^{2\pi i/3}q^2;q^4)_\infty(-e^{-2\pi i/3}q^2;q^4)_\infty(q^4;q^4)_\infty \\
&= \frac{(-q^6;q^{12})_\infty(q^4;q^4)_\infty}{(-q^2;q^4)_\infty}.
\end{aligned}$$

Using the calculations above in Entry 5.3.5, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-q^3;q^6)_n}{(q^2;q^2)_{2n}}q^{n^2} &= \frac{f(e^{2\pi i/3}q^2,e^{-2\pi i/3}q^2)}{\psi(-q)}\\ &= \frac{(-q^6;q^{12})_\infty(q^4;q^4)_\infty}{\psi(-q)(-q^2;q^4)_\infty}\\ &= \frac{\psi(-q^2)}{\psi(-q)}(-q^6;q^{12})_\infty,\end{aligned}$$

by (1.4.10). □

Entry 5.3.9 (p. 33, corrected). *We have*

$$\sum_{n=0}^{\infty}\frac{(q^3;q^6)_n q^{n^2}}{(q;q^2)_n^2(q^4;q^4)_n} = \frac{\psi(q^2)}{\psi(-q)}(q^6;q^{12})_\infty.$$

Proof. In Entry 5.3.5, set $a = e^{\pi i/3}$. Using calculations very similar to those of the previous proof, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(q^3;q^6)_n q^{n^2}}{(q;q^2)_n^2(q^4;q^4)_n} &= \frac{f(e^{\pi i/3}q^2,e^{-\pi i/3}q^2)}{\psi(-q)}\\ &= \frac{(-e^{\pi i/3}q^2;q^4)_\infty(-e^{-\pi i/3}q^2;q^4)_\infty(q^4;q^4)_\infty}{\psi(-q)}\\ &= \frac{(q^6;q^{12})_\infty(q^4;q^4)_\infty}{\psi(-q)(q^2;q^4)_\infty}\\ &= \frac{\psi(q^2)}{\psi(-q)}(q^6;q^{12})_\infty,\end{aligned}$$

by (1.4.10). □

A. Sills has pointed out that Entry 5.3.9 is, in fact, a specialization of the q-analogue of Bailey's theorem [15, p. 526, equation (1.9)]; namely, one replaces q by q^2 and then sets $b = qe^{2\pi i/3}$ and $c = q$ to obtain Entry 5.3.9. In addition, Sills notes that Entry 5.3.9 can be obtained by taking equation (109) minus q times equation (110) (corrected) in Slater's compendium of Rogers–Ramanujan-type identities [262].

Entry 5.3.10 (p. 33). *Formally, for* $a \neq 0$,

$$\sum_{n=0}^{\infty}\frac{(-1)^n(-aq;q^2)_n(-q/a;q^2)_n}{(q^4;q^4)_n} = \frac{f(-aq,-q/a)}{2\psi(q^2)}. \tag{5.3.10}$$

The series on the left-hand side of (5.3.10) is divergent for all q. However, if we set $\rho_1 = q$ and $\rho_2 = -q$ in (5.2.3), we see that term by term, the left side of the resulting identity coincides with the left side of (5.3.10).

Proof. We interpret the left-hand side of (5.3.10) as

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1)^n(-aq;q^2)_n(-q/a;q^2)_n}{(q^4;q^4)_n} \\
&\quad= \lim_{\substack{\rho_1\to q\\ \rho_2\to -q}} \sum_{n=0}^{\infty} \frac{(\rho_1;q^2)_n(\rho_2;q^2)_n(-aq;q^2)_n(-a^{-1}q;q^2)_n}{(q;q^2)_n(-q;q^2)_n(q^4;q^4)_n}\left(\frac{q^2}{\rho_1\rho_2}\right)^n \\
&\quad= \lim_{\substack{\rho_1\to q\\ \rho_2\to -q}} \frac{(q^2/\rho_1;q^2)_\infty(q^2/\rho_2;q^2)_\infty}{(q^2;q^2)_\infty(q^2/(\rho_1\rho_2);q^2)_\infty} \\
&\qquad\times\left(1+\sum_{N=1}^{\infty}\frac{(\rho_1;q^2)_N(\rho_2;q^2)_N(a^N+a^{-N})}{(q^2/\rho_1;q^2)_N(q^2/\rho_2;q^2)_N}\left(\frac{1}{\rho_1\rho_2}\right)^N q^{N^2+2N}\right) \\
&\quad= \frac{(q^2;q^4)_\infty}{2(q^4;q^4)_\infty}\sum_{N=-\infty}^{\infty}(-1)^N a^N q^{N^2} \\
&\quad= \frac{f(-aq,-q/a)}{2\psi(q^2)},
\end{aligned}$$

by (1.4.8) and (1.4.10). □

Another formal argument yielding Entry 5.3.10 was devised by Padmavathamma [225].

We have seen in this chapter many beautiful representations for quotients of theta functions by q-series. In particular, several representations for $f(q^a,q^b)/\psi(q)$ for various a and b have been proved. D. Bowman, J. McLaughlin, and A. Sills [94] have recently found a q-series representation for $f(q,q^4)/\psi(q)$ that Ramanujan apparently did not discover.

5.4 Corollaries of (5.2.4) and Related Results

Entry 5.4.1 (p. 34). *For $a \neq 0,-1$,*

$$\begin{aligned}
&\frac{1}{a^{1/2}+a^{-1/2}}\sum_{n=0}^{\infty}(-1)^n(a^{n+1/2}+a^{-n-1/2})q^{n(n+1)} \\
&\qquad= \sum_{n=0}^{\infty}\frac{(-1)^n(-aq)_n(-q/a)_n}{(q^{n+1})_{n+1}}q^{n(n+1)/2}.
\end{aligned}$$

First Proof of Entry 5.4.1. In (5.2.4), let $x=a$ and $\rho_1=q$, and then let $\rho_2\to\infty$. After simplification, we find that

$$(1+a)\sum_{n=0}^{\infty}\frac{(-1)^n(-aq)_n(-q/a)_n}{(q^{n+1})_{n+1}}q^{n(n+1)/2}=\sum_{n=0}^{\infty}(-1)^n\left(a^{n+1}+a^{-n}\right)q^{n(n+1)}. \tag{5.4.1}$$

If we now divide both sides of this last equality by $(1+a)$ and then multiply the numerator and denominator on the right-hand side by $a^{-1/2}$, we obtain the desired result. □

Padmavathamma [225] has also proved Entry 5.4.1 as well as the following entry. Since her proof of Entry 5.4.1 is quite different, we provide it here. We need two preliminary results: Lemma 5.3.1 and a corollary of Heine's transformation (1.1.3).

Lemma 5.4.1. *For* $0<|b|<1$,

$$\sum_{n=0}^{\infty}\frac{(b)_n z^n q^{n(n-1)/2}}{(c)_n(q)_n}=\frac{(b)_\infty(-z)_\infty}{(c)_\infty}\sum_{m=0}^{\infty}\frac{(c/b)_m b^m}{(-z)_m(q)_m}.$$

First Proof of Entry 5.4.1. In Heine's transformation (1.1.3), replace a by a/d, set $t=dz$, let $a=1$, and let d tend to 0. □

Second Proof of Entry 5.4.1. Setting $i=n$ and $j=n+1$, replacing q^2 by q, and then replacing a by $a\sqrt{q}$ in Lemma 5.3.1, we find that

$$(-aq)_n(-1/a)_{n+1}=\sum_{m=0}^{n}\frac{(q)_{2n+1}a^m q^{m(m+1)/2}}{(q)_{n-m}(q)_{m+n+1}}+\sum_{m=1}^{n+1}\frac{(q)_{2n+1}a^{-m}q^{m(m-1)/2}}{(q)_{m+n}(q)_{n-m+1}}. \tag{5.4.2}$$

For brevity, set

$$c(m,n):=\frac{(q)_{2n+1}q^{m(m+1)/2}}{(q)_{n-m}(q)_{m+n+1}}.$$

Observe that $c(m,n)=c(-m-1,n)$. Thus, we can write (5.4.2) in the abbreviated form

$$(-aq)_n(-1/a)_{n+1}=\sum_{m=0}^{n}c(m,n)\left(a^m+a^{-m-1}\right). \tag{5.4.3}$$

Multiplying both sides of (5.4.1) by $1+1/a$, we find that it suffices to prove that

$$\sum_{n=0}^{\infty}(-1)^n\left(a^n+a^{-n-1}\right)q^{n(n+1)}=\sum_{n=0}^{\infty}\frac{(-1)^n(-aq)_n(-1/a)_{n+1}}{(q^{n+1})_{n+1}}q^{n(n+1)/2}. \tag{5.4.4}$$

Using (5.4.3) in (5.4.4) and inverting the order of summation, we now find that it is sufficient to show that

$$\sum_{n=0}^{\infty}(-1)^n\left(a^n+a^{-n-1}\right)q^{n(n+1)}$$
$$=\sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\frac{(-1)^n c(m,n)(a^m+a^{-m-1})}{(q^{n+1})_{n+1}}q^{n(n+1)/2}.$$

Hence, it suffices to prove that for each nonnegative integer m,

$$\sum_{n=m}^{\infty} \frac{(-1)^n c(m,n) q^{n(n+1)/2}}{(q^{n+1})_{n+1}} = (-1)^m q^{m(m+1)}. \tag{5.4.5}$$

Now, setting $t = q^{m+1}$ below, we find that

$$\begin{aligned}
\sum_{n=m}^{\infty} \frac{(-1)^n c(m,n) q^{n(n+1)/2}}{(q^{n+1})_{n+1}} &= (-1)^m \sum_{k=0}^{\infty} \frac{(-1)^k c(m,m+k) q^{(m+k)(m+k+1)/2}}{(q^{m+k+1})_{m+k+1}} \\
&= (-1)^m q^{m(m+1)} \sum_{k=0}^{\infty} \frac{(-1)^k (q)_{2m+2k+1} q^{mk} q^{k(k+1)/2}}{(q)_k (q)_{2m+k+1} (q^{m+k+1})_{m+k+1}} \\
&= \frac{(-1)^m q^{m(m+1)}}{(q^{m+1})_{m+1}} \sum_{k=0}^{\infty} \frac{(-1)^k (q^{m+1})_k q^{(m+1)k} q^{k(k-1)/2}}{(q)_k (q^{2m+2})_k} \\
&= \frac{(-1)^m q^{m(m+1)}}{(t)_{m+1}} \sum_{k=0}^{\infty} \frac{(-1)^k (t)_k t^k q^{k(k-1)/2}}{(q)_k (t^2)_k}.
\end{aligned}$$

Apply Lemma 5.4.1 with $b = t$, $z = -t$, and $c = t^2$ in the last line above and then invoke (1.2.3) to deduce that

$$\begin{aligned}
\sum_{n=m}^{\infty} \frac{(-1)^n c(m,n) q^{n(n+1)/2}}{(q^{n+1})_{n+1}} &= \frac{(-1)^m q^{m(m+1)}}{(t)_{m+1}} \frac{(t)_\infty^2}{(t^2)_\infty} \sum_{m=0}^{\infty} \frac{t^m}{(q)_m} \\
&= (-1)^m q^{m(m+1)} (t)_\infty \frac{1}{(t)_\infty} = (-1)^m q^{m(m+1)},
\end{aligned}$$

because, since $t = q^{m+1}$,

$$\frac{(t)_\infty}{(t^2)_\infty} = (t)_{m+1}.$$

□

The next entry was first discovered by F.J. Dyson and proved by Bailey [40, p. 434, equation (E2)].

Entry 5.4.2 (p. 34). *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q^3;q^3)_n}{(q;q)_{2n+1}} q^{n(n+1)/2} = f(q^{12}, q^6).$$

Proof. In Entry 5.4.1, set $a = -e^{2\pi i/3} = e^{-\pi i/3}$ and observe that

$$(-1)^n \frac{a^{n+1/2} + a^{-n-1/2}}{a^{1/2} + a^{-1/2}} = (-1)^n \frac{a^{n+1} + a^{-n}}{a+1} = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 3, \\ 0, & \text{if } n \equiv 1 \pmod 3, \\ -1, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Hence, from Entry 5.4.1,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n(q^3;q^3)_n}{(q;q)_{2n+1}}q^{n(n+1)/2} &= \sum_{\substack{n=0\\ n\equiv 0\,(\mathrm{mod}\,3)}}^{\infty} q^{n(n+1)} - \sum_{\substack{n=0\\ n\equiv 2\,(\mathrm{mod}\,3)}}^{\infty} q^{n(n+1)}\\ &= \sum_{n=0}^{\infty} q^{9n^2+3n} - \sum_{n=0}^{\infty} q^{9n^2+15n+6}\\ &= \sum_{n=0}^{\infty} q^{9n^2+3n} - \sum_{j=-1}^{-\infty} q^{9j^2+3j}\\ &= f(q^{12},q^6),\end{aligned}$$

where in the antepenultimate line we set $n=-j-1$. $\square$

Entry 5.4.3 (p. 4). *Formally,*

$$\left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty}\frac{(-1)^n(-aq;q)_n(-q/a;q)_n}{(q;q^2)_{n+1}} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(a^n+a^{-n-1})q^{n(n+1)/2}. \tag{5.4.6}$$

As in the case of Entry 5.3.10, the left-hand side of (5.4.6) does not converge for any value of q. We show that the left-hand side of (5.2.4) reduces term by term to the left-hand side of (5.4.6).

Proof. Setting $x=1/a$ in (5.2.4), we see that

$$\begin{aligned}&\left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty}\frac{(-1)^n(-aq;q)_n(-q/a;q)_n}{(q;q^2)_{n+1}}\\ &\quad= \lim_{\substack{\rho_1\to q\\ \rho_2\to -q}}\sum_{n=0}^{\infty}\frac{(-1/a)_{n+1}(-aq)_n(\rho_1)_n(\rho_2)_n}{(q)_{2n+1}}\left(\frac{q^2}{\rho_1\rho_2}\right)^n\\ &\quad= \lim_{\substack{\rho_1\to q\\ \rho_2\to -q}}\frac{(q^2/\rho_1)_\infty(q^2/\rho_2)_\infty}{(q)_\infty(q^2/(\rho_1\rho_2))_\infty}\\ &\qquad\times\sum_{n=0}^{\infty}\frac{(\rho_1)_n(\rho_2)_n}{(q^2/\rho_1)_n(q^2/\rho_2)_n}(a^{-n-1}+a^n)\left(\frac{q^2}{\rho_1\rho_2}\right)^n q^{n(n+1)/2}\\ &\quad= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(a^n+a^{-n-1})q^{n(n+1)/2}.\end{aligned}$$

$\square$

Entry 5.4.4 (p. 15). *For* $a\neq 0$,

$$\left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty}\frac{(aq;q^2)_n(q/a;q^2)_n}{(-q;q)_{2n+1}}q^n = \sum_{n=0}^{\infty}(-1)^n(a^n+a^{-n-1})q^{n^2+n}. \tag{5.4.7}$$

While this entry is certainly appropriate to this section, we have not found a proof that is a direct corollary of the identities in Section 5.2.

Proof. In (1.7.3), set $n = 2N$, $a = 1$, and replace q by q^2. Then set $b = aq^{-2N+1}$ and replace the index of summation by $N - j$. This yields, after enormous simplification,

$$\begin{aligned}(aq;q^2)_N(a^{-1}q;q^2)_N &= \sum_{j=-\infty}^{\infty} \begin{bmatrix} 2N \\ N-j \end{bmatrix}_{q^2} (-1)^j a^{-j} q^{j^2} \\ &= \sum_{j=-\infty}^{\infty} \begin{bmatrix} 2N \\ N-j \end{bmatrix}_{q^2} (-1)^j a^{j} q^{j^2}, \end{aligned} \tag{5.4.8}$$

where in the last step we replaced j by $-j$ and used the symmetry of the q-binomial coefficients. Hence, using (5.4.8) and the q-analogue of Pascal's formula for binomial coefficients, we find that

$$\begin{aligned} S &:= \left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty} \frac{(aq;q^2)_n(q/a;q^2)_n}{(-q;q)_{2n+1}} q^n \\ &= \left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_{2n+1}} \sum_{j=-\infty}^{\infty} \begin{bmatrix} 2n \\ n-j \end{bmatrix}_{q^2} (-1)^j a^j q^{j^2} \\ &= \sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_{2n+1}} \Bigg(\sum_{j=-\infty}^{\infty} \begin{bmatrix} 2n \\ n-j \end{bmatrix}_{q^2} (-1)^j a^j q^{j^2} \\ &\qquad - \sum_{j=-\infty}^{\infty} \begin{bmatrix} 2n \\ n-j-1 \end{bmatrix}_{q^2} (-1)^j a^j q^{(j+1)^2} \Bigg) \\ &= \sum_{j=-\infty}^{\infty} (-1)^j a^j q^{j^2} \sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_{2n+1}} \left(\begin{bmatrix} 2n \\ n-j \end{bmatrix}_{q^2} - q^{2j+1} \begin{bmatrix} 2n \\ n-j-1 \end{bmatrix}_{q^2} \right). \end{aligned} \tag{5.4.9}$$

We first examine the inner sum on the far right side of (5.4.9) for $j \geq 0$. To that end,

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_{2n+1}} \left(\begin{bmatrix} 2n \\ n-j \end{bmatrix}_{q^2} - q^{2j+1} \begin{bmatrix} 2n \\ n-j-1 \end{bmatrix}_{q^2} \right) \\ &\qquad = \sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_{2n+1}} \frac{(q^2;q^2)_{2n}\left((1-q^{2n+2j+2}) - q^{2j+1}(1-q^{2n-2j})\right)}{(q^2;q^2)_{n-j}(q^2;q^2)_{n+j+1}} \\ &\qquad = \sum_{n=0}^{\infty} \frac{(q^2;q^2)_{2n}(1+q^{2n+1})(1-q^{2j+1})}{(-q;q)_{2n+1}(q^2;q^2)_{n-j}(q^2;q^2)_{n+j+1}} q^n \\ &\qquad = (1-q^{2j+1}) \sum_{n=0}^{\infty} \frac{(q;q)_{2n}}{(q^2;q^2)_{n-j}(q^2;q^2)_{n+j+1}} q^n \end{aligned}$$

$$
\begin{aligned}
&= (1-q^{2j+1})\sum_{n=0}^{\infty}\frac{(q;q)_{2n+2j}}{(q^2;q^2)_n(q^2;q^2)_{n+2j+1}}q^{n+j}\\
&= \frac{(q;q)_{2j+1}q^j}{(q^2;q^2)_{2j+1}}\sum_{n=0}^{\infty}\frac{(q^{2j+1};q^2)_n(q^{2j+2};q^2)_n}{(q^2;q^2)_n(q^{4j+4};q^2)_n}q^n\\
&= \frac{(q;q)_{2j+1}q^j}{(q^2;q^2)_{2j+1}}\frac{(q^{2j+3};q^2)_\infty(q^{2j+2};q^2)_\infty}{(q^{4j+4};q^2)_\infty(q;q^2)_\infty}\\
&= \frac{(q;q)_\infty q^j}{(q;q)_\infty}\\
&= q^j, \qquad (5.4.10)
\end{aligned}
$$

where in the antepenultimate line we applied (1.2.9) with q replaced by q^2, and then set $a = q^{2j+1}$, $b = q^{2j+2}$, $c = q^{4j+4}$, and $t = q$. Thus, the contribution of (5.4.10) to the sum S of (5.4.9), i.e., the terms for $j \geq 0$, is equal to

$$\sum_{j=0}^{\infty}(-1)^j a^j q^{j^2+j}, \qquad (5.4.11)$$

which indeed does give us part of the right-hand side of (5.4.7).

Now if we replace a by a^{-1} in the left-hand side of (5.4.7) and then multiply by a^{-1}, we see that the left-hand side of (5.4.7) is invariant. Consequently, the coefficient of a^{-j-1} on the right-hand side of (5.4.7) is the same as the coefficient of a^j. Hence, using (5.4.11) and the observation that we just made in (5.4.9), we conclude that

$$S = \left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty}\frac{(aq;q^2)_n(q/a;q^2)_n}{(-q;q)_{2n+1}}q^n = \sum_{j=0}^{\infty}(-1)^j(a^j + a^{-j-1})q^{j^2+j},$$

as desired. □

If we divide both sides of (5.4.7) by $(1 + 1/a)$ and let $a \to -1$, we deduce the identity

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^n}{(-q^2;q^2)_n(1+q^{2n+1})} = \sum_{j=0}^{\infty}(2j+1)q^{j^2+j},$$

which should be compared with Jacobi's identity

$$(q^2;q^2)_\infty^3 = \sum_{j=0}^{\infty}(-1)^j(2j+1)q^{j^2+j}.$$

6

Partial Theta Functions

6.1 Introduction

In the lost notebook we find a number of identities involving sums of the form

$$\sum_{n=0}^{\infty} z^n q^{An^2+Bn}. \tag{6.1.1}$$

Inasmuch as this sum consists of the terms with positive index taken from the classical Jacobi theta function

$$\sum_{n=-\infty}^{\infty} z^n q^{An^2+Bn} = (-zq^{A+B};q^{2A})_\infty(-q^{A-B}/z;q^{2A})_\infty(q^{2A};q^{2A})_\infty, \tag{6.1.2}$$

we have chosen to name the series in (6.1.1) *partial theta functions*. We have chosen the designation partial theta functions, in contrast with L.J. Rogers's "false theta functions" discussed in Chapters 9 and 11 of our first volume [31, pp. 227–239, 256–259]. The false theta functions are instances of the full series in (6.1.2) with z specialized to $\pm q^a$ for some real number a but with a sign pattern inconsistent with a specialization of (6.1.2). In general, then, partial theta functions are not the same as false theta functions; however, most false theta functions are a sum of two specializations of partial theta functions. Formulas such as those in Entry 6.5.1 might be regarded purely as false theta function identities; however, their proofs rely on earlier results in this chapter. Consequently, it is natural to include them here.

The majority of results in this chapter first appeared in [21] and [28]. Subsequently, R.P. Agarwal [4] made a major contribution in placing the main lemma in [21] within the standard hierarchy of the theory of q-hypergeometric series. Most recently, S.O. Warnaar [273] has discovered a truly beautiful identity connecting the sum of two independent partial theta functions. Warnaar's work elucidates some of the more recondite partial theta function identities.

As in other chapters, such as Chapter 9 of [31] on the Rogers–Fine identity, we need central theorems, such as Theorem 6.2.1 in Section 6.2 and Warnaar's

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theorem in Section 6.6, which are not stated by Ramanujan but which yield most of the results in this chapter. After proving Theorem 6.2.1 in Section 6.2, we devote Section 6.3 to deducing results on partial theta functions. Section 6.4 is devoted to the examination of several entries on a particular partial theta function. Various extensions of Euler's identity are the focus of Section 6.5. In Section 6.6, we prove an expansion theorem for the product of partial theta functions [38], and from this result we deduce Warnaar's theorem. We then consider implications of Warnaar's theorem. Finally, after all of these developments, there remains one recalcitrant formula of Ramanujan, which is proved in Section 6.6.

6.2 A General Identity

After [21] appeared with its laborious proof of a central lemma, Agarwal [4] showed how the result is implied by one of D.B. Sears's three-term ${}_3\phi_2$ relations [256]. We follow Agarwal's elegant account.

Theorem 6.2.1. *For any parameters A, B, $a \neq 0$, and b,*

$$\sum_{n=0}^{\infty} \frac{(B)_n(-Abq)_n}{(-aq)_n(-bq)_n} q^n = -\frac{(B)_\infty(-Abq)_\infty}{a(-aq)_\infty(-bq)_\infty} \sum_{m=0}^{\infty} \frac{(1/A)_m}{(-B/a)_{m+1}} \left(\frac{Abq}{a}\right)^m + (1+b) \sum_{m=0}^{\infty} \frac{(-1/a)_{m+1}(-ABq/a)_m}{(-B/a)_{m+1}(Abq/a)_{m+1}} (-b)^m.$$

Proof. The relevant identity of Sears [256, p. 173, equations II(a)–(c)] is given by

$$(e)_\infty(f)_\infty \left(\frac{ef}{abc}\right)_\infty \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(q)_n(e)_n(f)_n} \left(\frac{ef}{abc}\right)^n = \frac{(b)_\infty(e/a)_\infty(f/a)_\infty(ef/(bc))_\infty}{(b/a)_\infty} \sum_{n=0}^{\infty} \frac{(e/b)_n(f/b)_n(a)_n}{(q)_n(aq/b)_n(ef/(bc))_n} q^n + \frac{(a)_\infty(e/b)_\infty(f/b)_\infty(ef/(ac))_\infty}{(a/b)_\infty} \sum_{n=0}^{\infty} \frac{(e/a)_n(f/a)_n(b)_n}{(q)_n(bq/a)_n(ef/(ac))_n} q^n.$$

Replacing a by q, we deduce that, after simplification,

$$\frac{(1-b/q)(1-ef/(bcq))}{(1-e/q)(1-f/q)} \sum_{n=0}^{\infty} \frac{(b)_n(c)_n}{(e)_n(f)_n} \left(\frac{ef}{bcq}\right)^n = \sum_{n=0}^{\infty} \frac{(e/b)_n(f/b)_n}{(q^2/b)_n(ef/(bc))_n} q^n + \frac{(1-b/q)(e/b)_\infty(f/b)_\infty(q)_\infty(ef/(cq))_\infty}{(ef/(bc))_\infty(e/q)_\infty(f/q)_\infty(q/b)_\infty} \sum_{n=0}^{\infty} \frac{(e/q)_n(f/q)_n}{(q)_n(ef/(cq))_n} q^n$$

$$= \sum_{n=0}^{\infty} \frac{(e/b)_n (f/b)_n}{(q^2/b)_n (ef/(bc))_n} q^n$$
$$+ \frac{(1-b/q)(e/b)_\infty (f/b)_\infty (e)_\infty}{(ef/(bc))_\infty (e/q)_\infty (q/b)_\infty} \sum_{n=0}^{\infty} \frac{(e/c)_n}{(e)_n} \left(\frac{f}{q}\right)^n,$$

where we have transformed the second sum on the right-hand side using Heine's transformation (1.1.3). Now replace b by $-q/a$, and then replace f by Abq^2/a, e by $-Bq/a$, and c by $-ABq/a$. Theorem 6.2.1 now follows. □

6.3 Consequences of Theorem 6.2.1

Entry 6.3.1 (p. 40). *We have, for $a \neq 0$ and any number b,*

$$\sum_{n=0}^{\infty} \frac{q^n}{(-aq)_n(-bq)_n} = \frac{1}{(-aq)_\infty(-bq)_\infty} \sum_{m=0}^{\infty} (-1)^{m+1} a^{-m-1} b^m q^{m(m+1)/2}$$
$$+ \left(1+\frac{1}{a}\right) \sum_{m=0}^{\infty} \frac{(-1)^m a^{-m} b^m q^{m(m+1)/2}}{(-bq)_m}.$$

Proof. In Theorem 6.2.1, set $A = B = 0$. This yields

$$\sum_{n=0}^{\infty} \frac{q^n}{(-aq)_n(-bq)_n} = \frac{1}{(-aq)_\infty(-bq)_\infty} \sum_{m=0}^{\infty} (-1)^{m+1} a^{-m-1} b^m q^{m(m+1)/2}$$
$$+ (1+b) \sum_{m=0}^{\infty} (-1/a)_{m+1} (-b)^m$$
$$= \frac{1}{(-aq)_\infty(-bq)_\infty} \sum_{m=0}^{\infty} (-1)^{m+1} a^{-m-1} b^m q^{m(m+1)/2}$$
$$+ \left(1+\frac{1}{a}\right) \sum_{m=0}^{\infty} \frac{(-1)^m a^{-m} b^m q^{m(m+1)/2}}{(-bq)_m},$$

where we applied (1.2.9) with a, b, and t replaced by $-q/a$, q, and $-b$, respectively, and then let $c \to 0$. □

B. Kim [189] provided a beautiful bijective proof of Entry 6.3.1 and so successfully solved a problem posed by I. Pak [226].

Entry 6.3.2 (p. 37). *For $a \neq 0$,*

$$\sum_{n=0}^{\infty} \frac{q^n}{(-aq)_n(-q/a)_n} = (1+a) \sum_{n=0}^{\infty} a^{3n} q^{n(3n+1)/2} (1 - a^2 q^{2n+1})$$
$$- \frac{1}{(-aq)_\infty(-q/a)_\infty} \sum_{n=0}^{\infty} (-1)^n a^{2n+1} q^{n(n+1)/2}.$$

Proof. In Entry 6.3.1, replace a by $1/a$ and then replace b by a to find that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{q^n}{(-aq)_n(-q/a)_n} &= -\frac{1}{(-aq)_\infty(-q/a)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{2n+1}q^{n(n+1)/2}\\ &\quad+(1+a)\sum_{m=0}^{\infty}\frac{(-1)^m a^{2m}q^{m(m+1)/2}}{(-aq)_m}\\ &= -\frac{1}{(-aq)_\infty(-q/a)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{2n+1}q^{n(n+1)/2}\\ &\quad+(1+a)\sum_{n=0}^{\infty}a^{3n}q^{n(3n+1)/2}(1-a^2q^{2n+1}),\end{aligned}$$

by Entry 9.4.1 of [31]. □

Entry 6.3.3 (p. 40). *For $ab \neq 0$,*

$$\begin{aligned}\left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n a^n b^{-n}q^{n(n+1)/2}}{(-aq)_n} - \left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty}\frac{(-1)^n a^{-n} b^{n}q^{n(n+1)/2}}{(-bq)_n}\\ = \left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b)_\infty(bq/a)_\infty(q)_\infty}{(-aq)_\infty(-bq)_\infty}.\end{aligned}$$

Proof. Subtract Entry 6.3.1 from Entry 6.3.1 with a and b interchanged. Consequently,

$$\begin{aligned}0 &= \left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n a^n b^{-n}q^{n(n+1)/2}}{(-aq)_n} - \left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty}\frac{(-1)^n a^{-n} b^{n}q^{n(n+1)/2}}{(-bq)_n}\\ &\quad-\frac{1}{(-aq)_\infty(-bq)_\infty}\sum_{n=0}^{\infty}(-1)^n\left(a^n b^{-n-1}-a^{-n-1}b^n\right)q^{n(n+1)/2}\\ &= \left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n a^n b^{-n}q^{n(n+1)/2}}{(-aq)_n} - \left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty}\frac{(-1)^n a^{-n} b^{n}q^{n(n+1)/2}}{(-bq)_n}\\ &\quad-\frac{1}{(-aq)_\infty(-bq)_\infty}\sum_{n=-\infty}^{\infty}(-1)^n a^n b^{-n-1}q^{n(n+1)/2}.\end{aligned}$$

Now from (1.4.8),

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(-1)^n a^n b^{-n-1}q^{n(n+1)/2} &= \frac{1}{b}f(-aq/b,-b/a)\\ &= \frac{1}{b}(aq/b)_\infty(b/a)_\infty(q)_\infty\\ &= \frac{1}{b}\left(1-\frac{b}{a}\right)(aq/b)_\infty(bq/a)_\infty(q)_\infty\\ &= \left(\frac{1}{b}-\frac{1}{a}\right)(aq/b)_\infty(bq/a)_\infty(q)_\infty.\end{aligned}$$

Thus, using this last representation in the foregoing identity, we deduce that

$$0 = \left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n a^n b^{-n} q^{n(n+1)/2}}{(-aq)_n} - \left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty}\frac{(-1)^n a^{-n} b^{n} q^{n(n+1)/2}}{(-bq)_n}$$
$$- \left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b)_\infty (bq/a)_\infty (q)_\infty}{(-aq)_\infty(-bq)_\infty}.$$

This is equivalent to the desired result. □

Early in the twenty-first century, several further proofs of Entry 6.3.3 were independently given, sometimes without knowledge that the theorem can be found in the lost notebook. D.D. Somashekara and S.N. Fathima [264] used Ramanujan's ${}_1\psi_1$ summation theorem and Heine's transformation to establish an equivalent version of Entry 6.3.3. S. Bhargava, Somashekara, and Fathima [83] provided another proof, which is a slight variation of that in [264]. Proceeding from scratch and using the q-binomial theorem, T. Kim, Somashekara, and Fathima [193] gave a much different proof of Entry 6.3.3. P.S. Guruprasad and N. Pradeep [170] also devised a proof using the q-binomial theorem. C. Adiga and N. Anitha [2] devised a proof of Entry 6.3.3 along the lines of M.E.H. Ismail's proof [184] of Ramanujan's ${}_1\psi_1$ summation formula. Berndt, S.H. Chan, B.P. Yeap, and A.J. Yee [71] found three proofs of Entry 6.3.3. Their first, using the second iterate of Heine's transformation and the ${}_1\psi_1$ summation theorem, is similar to that of Somashekara and Fathima [264]. Their second employs the Rogers–Fine identity. Their third is combinatorial and so completely different from other proofs. As a corollary of their work, they derive a two-variable generalization of the quintuple product identity.

S.-Y. Kang [188] constructed a proof of Entry 6.3.3 along the lines of K. Venkatachaliengar's proof of the Ramanujan ${}_1\psi_1$ summation formula [3], [54, 32–34]. She also obtained a four-variable generalization of Entry 6.3.3 and a four-variable generalization of the quintuple product identity.

Theorem 6.3.1. *Suppose that $ab \neq 0$ and that c and d are any parameters, except that $c, d \neq -aq^{-n}, -bq^{-n}$, for any positive integer n. Let*

$$\rho_4(a,b;c;d) := \left(1+\frac{1}{b}\right)\sum_{n=0}^{\infty}\frac{(-1)^n (d,c,cd/(ab))_n (1+cdq^{2n}/b) a^n b^{-n} q^{n(n+1)/2}}{(-aq)_n(-c/b,-d/b)_{n+1}}.$$

Then

$$\rho_4(a,b;c;d) - \rho_4(a,b;d;c) = \left(\frac{1}{b}-\frac{1}{a}\right)\frac{(c,d,cd/(ab),aq/b,bq/a,q)_\infty}{(-d/a,-d/b,-c/a,-c/b,-aq,-bq)_\infty}.$$

Note that if $c = d = 0$, Theorem 6.3.1 reduces to Entry 6.3.3.

Z. Zhang [290] also found a three-variable generalization of Entry 6.3.3. An earlier attempt [289] to generalize Entry 6.3.3 evidently contained a mistake.

Theorem 6.3.2. *If a and b are not equal to 0, and aq, adq, bq, $bdq \neq -q^{-n}$, $n \geq 0$, then*

$$\left(1+\frac{1}{b}\right)(1+bd)\sum_{n=0}^{\infty}\frac{(-1)^n(abd)_n a^n b^{-n} q^{n(n+1)/2}}{(-aq,-adq)_n}$$
$$-\left(1+\frac{1}{a}\right)(1+ad)\sum_{n=0}^{\infty}\frac{(-1)^n(abd)_n a^{-n} b^{n} q^{n(n+1)/2}}{(-bq,-bdq)_n}$$
$$=\left(\frac{1}{b}-\frac{1}{a}\right)\frac{(aq/b,bq/a,abd,q)_\infty}{(-aq,-bq,-adq,-bdq)_\infty}.$$

Z.-G. Liu [205] has also generalized Entry 6.3.3 in the following theorem.

Theorem 6.3.3.

$$d\sum_{n=0}^{\infty}\frac{(q/bc,acdf;q)_n}{(ad,df;q)_{n+1}}(bd)^n - c\sum_{n=0}^{\infty}\frac{(q/bd,acdf;q)_n}{(ac,cf;q)_{n+1}}(bc)^n$$
$$=\frac{(q,qd/c,c/d,abcd,acdf,bcdf;q)_\infty}{(ac,ad,cf,df,bc,bd;q)_\infty}. \tag{6.3.1}$$

If we set $b=f=0$ in (6.3.1), we deduce Entry 6.3.3. Taking $b=0$, we obtain Zhang's extension in Theorem 6.3.2.

Lastly, we remark that W. Chu and W. Zhang [130] have not only extended the results of Kang, but they have even extended Andrews's original result [21, Theorem 6] as well.

Although we have mentioned only a few applications of Entry 6.3.3 and the two generalizations recorded above, all of the papers that we have cited contain applications of the main theorems. In particular, some offer applications to sums of squares.

Entry 6.3.4 (p. 37). *If $a \neq 0$, then*

$$\sum_{n=0}^{\infty}\frac{q^{2n+1}}{(-aq;q^2)_{n+1}(-q/a;q^2)_{n+1}} = \sum_{n=0}^{\infty}a^{3n+1}q^{3n^2+2n}(1-aq^{2n+1})$$
$$-\frac{1}{(-aq;q^2)_\infty(-q/a;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{2n+1}q^{n(n+1)}.$$

Proof. In Theorem 6.2.1, replace q by q^2, then set $A=B=0$, and finally replace a and b by q/a and aq, respectively. Upon multiplying both sides of the result by $q(1+aq)^{-1}(1+q/a)^{-1}$, we find that

$$\sum_{n=0}^{\infty}\frac{q^{2n+1}}{(-aq;q^2)_{n+1}(-q/a;q^2)_{n+1}}$$
$$=-\frac{1}{(-aq;q^2)_\infty(-q/a;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{2n+1}q^{n(n+1)}$$

$$+ a\sum_{m=0}^{\infty}(-aq;q^2)_m(-aq)^m. \tag{6.3.2}$$

To conclude this proof, we apply (1.2.9) with q replaced by q^2, and then with a, b, and t replaced by $-aq$, q^2, and $-aq$, respectively. Then we let c tend to 0. Accordingly,

$$\begin{aligned}a\sum_{m=0}^{\infty}(-aq;q^2)_m(-aq)^m &= \sum_{m=0}^{\infty}\frac{(-1)^m a^{2m+1}q^{m(m+1)}}{(-aq;q^2)_{m+1}}\\ &= \sum_{n=0}^{\infty}a^{3n+1}q^{3n^2+2n}(1-aq^{2n+1}),\end{aligned}$$

by Entry 9.5.1 of [31]. Using the identity above in (6.3.2), we complete the proof. □

Entry 6.3.5 (p. 5). *For any complex number a,*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(aq;q^2)_n q^n}{(-q;q)_n} &= 2\sum_{n=0}^{\infty}(-1)^n a^n q^{n(n+1)}\\ &\quad - (q;q^2)_\infty(aq;q^2)_\infty\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)}}{(aq;q^2)_{n+1}}.\end{aligned}$$

Proof. We begin by applying (1.2.5) to the left-hand side of this entry with $t = q$, a replaced by aq, $b = q$, and $c = 0$. Hence, after simplification,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(aq;q^2)_n q^n}{(-q;q)_n} &= (q^2;q^2)_\infty(aq^2;q^2)_\infty\sum_{m=0}^{\infty}\frac{q^{2m}}{(q^2;q^2)_m(aq;q^2)_m}\\ &\quad + (q;q^2)_\infty(aq^3;q^2)_\infty\sum_{m=0}^{\infty}\frac{q^{2m+1}}{(q;q^2)_{m+1}(aq^3;q^2)_m}.\end{aligned} \tag{6.3.3}$$

Now by (1.7.6) with q replaced by q^2, we see that the first expression on the right-hand side of (6.3.3) has the representation

$$(q^2;q^2)_\infty(aq^2;q^2)_\infty\sum_{m=0}^{\infty}\frac{q^{2m}}{(q^2;q^2)_m(aq;q^2)_m} = \sum_{n=0}^{\infty}(-1)^n a^n q^{n(n+1)}.$$

Consequently, to conclude the proof of this entry, we must show that

$$\begin{aligned}&(q;q^2)_\infty(aq^3;q^2)_\infty\sum_{m=0}^{\infty}\frac{q^{2m+1}}{(q;q^2)_{m+1}(aq^3;q^2)_m}\\ &= \sum_{n=0}^{\infty}(-1)^n a^n q^{n(n+1)} - (q;q^2)_\infty(aq;q^2)_\infty\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)}}{(aq;q^2)_{n+1}}.\end{aligned} \tag{6.3.4}$$

But this last assertion follows immediately from Entry 6.3.1, as we now demonstrate. Replace q by q^2, replace a by $-q$, and then set $b=-aq$. Multiply both sides of the resulting identity by $q(q^3;q^2)_\infty(aq^3;q^2)_\infty$ and simplify to deduce (6.3.4). $\square$

Entry 6.3.6 (p. 8). *If $a\neq 0$ and $f(a,b)$ is defined by* (1.4.8), *then*

$$\left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{2n+1}}{(-aq;q^2)_{n+1}(-q/a;q^2)_{n+1}}$$
$$=\sum_{n=0}^{\infty}(-1)^n a^n q^{n(n+1)/2}-\frac{(q;q)_\infty}{f(aq,q/a)}\sum_{n=0}^{\infty}a^{3n}q^{3n^2+n}(1-a^2q^{4n+2}).$$

Proof. In Theorem 6.2.1, replace q by q^2 and then a by q/a; then set $B=0$, $b=aq$, and $A=-a^{-1}q^{-2}$; and lastly multiply the resulting identity by $q(1+1/a)(1+aq)^{-1}(1+q/a)^{-1}$. Consequently,

$$\left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{2n+1}}{(-aq;q^2)_{n+1}(-q/a;q^2)_{n+1}}$$
$$=-\frac{(q;q^2)_\infty}{(-aq;q^2)_\infty(-q/a;q^2)_\infty}\sum_{m=0}^{\infty}(-a;q^2)_{m+1}(-a)^m+\sum_{m=0}^{\infty}\frac{(-aq;q^2)_m(-aq)^m}{(-aq^2;q^2)_m}$$
$$=-\frac{(q;q)_\infty}{f(aq,q/a)}\sum_{m=0}^{\infty}(-a;q^2)_{m+1}(-a)^m+\sum_{m=0}^{\infty}\frac{(-aq;q^2)_m(-aq)^m}{(-aq^2;q^2)_m},$$

where we applied the Jacobi triple product identity (1.4.8).

Thus, to complete the proof of the entry, we must show that

$$\sum_{m=0}^{\infty}\frac{(-aq;q^2)_m(-aq)^m}{(-aq^2;q^2)_m}=\sum_{n=0}^{\infty}(-1)^n a^n q^{n(n+1)/2} \tag{6.3.5}$$

and

$$\sum_{m=0}^{\infty}(-a;q^2)_{m+1}(-a)^m=\sum_{n=0}^{\infty}a^{3n}q^{3n^2+n}(1-a^2q^{4n+2}). \tag{6.3.6}$$

To prove (6.3.5), invoke (1.7.1) with q replaced by q^2, and $\alpha=-aq$, $\beta=-aq^2$, and $\tau=-aq$ to find that

$$\sum_{m=0}^{\infty}\frac{(-aq;q^2)_m(-aq)^m}{(-aq^2;q^2)_m}=\sum_{n=0}^{\infty}a^{2n}q^{2n^2+n}(1-aq^{2n+1})$$
$$=\sum_{n=0}^{\infty}(-1)^n a^n q^{n(n+1)/2}.$$

To prove (6.3.6), apply (1.7.1) with q replaced by q^2 and $\alpha=-aq^2$, $\beta=0$, and $\tau=-a$, and lastly multiply both sides of the resulting identity by $(1+a)$. $\square$

Entry 6.3.7 (p. 2). *Recall that* $f(a,b)$ *and* $\psi(q)$ *are defined by* (1.4.8) *and* (1.4.10), *respectively. Then, for any complex number* $a \neq 0$,

$$\left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^{2n+1}}{(aq;q^2)_{n+1}(q/a;q^2)_{n+1}} = \frac{\psi(q)}{f(-aq,-q/a)}\sum_{n=0}^{\infty}(-a)^n q^{n(n+1)/2} - \sum_{n=0}^{\infty}(-a)^n q^{n(n+1)}.$$

Proof. In Theorem 6.2.1, replace q by q^2 and then a by $-q/a$; then set $B = -q$, $b = -aq$, and $A = -a^{-1}q^{-1}$; and finally multiply both sides of the result by $(1+1/a)q(1-aq)^{-1}(1-q/a)^{-1}$. Consequently,

$$\begin{aligned}\left(1+\frac{1}{a}\right)&\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^{2n+1}}{(aq;q^2)_{n+1}(q/a;q^2)_{n+1}} \\ &= \frac{(1+a)(-q;q)_\infty}{(aq;q^2)_\infty(q/a;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(-aq;q^2)_m(-aq)^m}{(-a;q^2)_{m+1}} \\ &\quad - \sum_{m=0}^{\infty}\frac{(aq;q^2)_m(q;q^2)_m(aq)^m}{(-aq^2;q^2)_m(-aq;q^2)_{m+1}}.\end{aligned}$$

By (6.3.5), we see that the first expression on the right-hand side of this latter result is equal to

$$\begin{aligned}&\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty(q^2;q^2)_\infty(aq;q^2)_\infty(q/a;q^2)_\infty}\sum_{n=0}^{\infty}(-a)^n q^{n(n+1)/2} \\ &\quad = \frac{\psi(q)}{f(-aq,-q/a)}\sum_{n=0}^{\infty}(-a)^n q^{n(n+1)/2},\end{aligned}$$

by (1.4.10) and the triple product identity (1.4.8). Therefore, to complete the proof of this entry, we must show that

$$S := \sum_{m=0}^{\infty}\frac{(aq;q^2)_m(q;q^2)_m(aq)^m}{(-aq^2;q^2)_m(-aq;q^2)_{m+1}} = \sum_{n=0}^{\infty}(-a)^n q^{n(n+1)}. \tag{6.3.7}$$

Now apply (1.2.1) with $k=2$, with a replaced by aq, and then with $b=q$, $c=-aq^2$, and $t=aq$. Thus,

$$\begin{aligned}S &= \frac{1}{1+aq}\sum_{m=0}^{\infty}\frac{(aq;q^2)_m(q;q)_{2m}(aq)^m}{(q^2;q^2)_m(-aq^2;q)_{2m}} \\ &= \frac{(q;q)_\infty(a^2q^2;q^2)_\infty}{(-aq;q)_\infty(aq;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(-aq;q)_m(aq;q^2)_m q^m}{(q;q)_m(a^2q^2;q^2)_m} \\ &= (q;q)_\infty(aq^2;q^2)_\infty\sum_{m=0}^{\infty}\frac{(\sqrt{aq};q)_m(-\sqrt{aq};q)_m q^m}{(q;q)_m(aq;q)_m}.\end{aligned}$$

We apply (1.2.1) once again, but now with a, b, and c replaced by $-\sqrt{aq}$, $\sqrt{aq}$, and aq, respectively, and with $t = q$. Multiplying both the numerator and denominator by $1 + \sqrt{aq}$, we find that

$$\begin{aligned} S &= \frac{(q;q)_\infty (aq^2;q^2)_\infty (aq;q^2)_\infty}{(aq;q)_\infty (q;q)_\infty} \sum_{m=0}^{\infty} \frac{(\sqrt{aq};q)_m (aq)^{m/2}}{(-\sqrt{aq};q)_{m+1}} \\ &= \sum_{n=0}^{\infty} (-a)^n q^{n(n+1)}, \end{aligned}$$

where we applied (1.7.1) with $\alpha = -\sqrt{aq}$, $\beta = q\sqrt{aq}$, and $\tau = \sqrt{aq}$. Hence, (6.3.7) has been demonstrated, and so the proof of Entry 6.3.7 is complete. ☐

Entry 6.3.8 (pp. 7, 13). *We have*

$$\sum_{n=0}^{\infty} \frac{(-q;q)_{2n} q^{2n+1}}{(q;q^2)_{n+1}^2} = \frac{1}{2(q;q^2)_\infty^3} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} - \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)}.$$

Proof. Set $a = 1$ in Entry 6.3.7 and note that

$$\frac{\psi(q)}{f(-q,-q)} = \frac{\psi(q)}{\varphi(-q)} = \frac{(q^2;q^2)_\infty (-q;q)_\infty}{(q;q^2)_\infty (q;q)_\infty} = \frac{1}{(q;q^2)_\infty^3},$$

by Euler's identity. The desired result now follows. ☐

Entry 6.3.9 (p. 29). *For $a \neq 0$,*

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n}}{(-aq^2;q^2)_n (-q^2/a;q^2)_n} &= (1+a) \sum_{n=0}^{\infty} (-a)^n q^{n(n+1)/2} \\ &\quad - \frac{a(q;q^2)_\infty}{(-aq^2;q^2)_\infty (-q^2/a;q^2)_\infty} \sum_{n=0}^{\infty} a^{3n} q^{3n^2+2n} (1 - aq^{2n+1}). \end{aligned}$$

Proof. In Theorem 6.2.1, replace q by q^2 and a by $1/a$. Then set $A = -a^{-1}q^{-1}$, $B = 0$, and $b = a$. Consequently,

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^{2n}}{(-aq^2;q^2)_n (-q^2/a;q^2)_n} \\ &\qquad = -\frac{a(q;q^2)_\infty}{(-aq^2;q^2)_\infty (-q^2/a;q^2)_\infty} \sum_{n=0}^{\infty} (-aq;q^2)_m (-aq)^m \\ &\qquad\quad + (1+a) \sum_{m=0}^{\infty} \frac{(-a;q^2)_{m+1} (-a)^m}{(-aq;q^2)_{m+1}}. \end{aligned}$$

To complete the proof of Entry 6.3.9, we must show that

$$\sum_{m=0}^{\infty}(-aq;q^2)_m(-aq)^m=\sum_{n=0}^{\infty}a^{3n}q^{3n^2+2n}(1-aq^{2n+1}) \tag{6.3.8}$$

and

$$\sum_{m=0}^{\infty}\frac{(-a;q^2)_{m+1}(-a)^m}{(-aq;q^2)_{m+1}}=\sum_{n=0}^{\infty}(-a)^nq^{n(n+1)/2}. \tag{6.3.9}$$

To prove (6.3.8), apply (1.7.1) with q replaced by q^2 and then with $\alpha=-aq$ and $\tau=-aq$. Letting $\beta\to 0$, we deduce (6.3.8). Invoke (1.7.1) once again, but now with $\alpha=-aq^2$, $\beta=-aq^3$, and $\tau=-a$. Multiplying both sides by $(1+a)/(1+aq)$, we find that

$$\begin{aligned}\sum_{m=0}^{\infty}\frac{(-a;q^2)_{m+1}(-a)^m}{(-aq;q^2)_{m+1}}&=\sum_{n=0}^{\infty}a^{2n}q^{2n^2+n}(1-aq^{2n+1})\\&=\sum_{n=0}^{\infty}(-a)^nq^{n(n+1)/2}.\end{aligned}$$

Thus, (6.3.9) has been shown, and hence the proof of Entry 6.3.9 is complete. □

Entry 6.3.10 (p. 29). *We have*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{2n}}{(-q^4;q^4)_n}=&\sum_{n=0}^{\infty}(-1)^nq^{n(n+1)/2}\\&+\frac{q\psi(q)}{(q^8;q^8)_\infty}\sum_{n=0}^{\infty}(-1)^nq^{12n^2+8n}(1+q^{8n+4}).\end{aligned}$$

Proof. In Entry 6.3.9, assume that q is real and replace q by $-q$. Setting $a=i$, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{2n}}{(-q^4;q^4)_n}=\mathrm{Re}\Bigg\{&(1+i)\sum_{n=0}^{\infty}(-i)^n(-q)^{n(n+1)/2}\\&-\frac{i(-q;q^2)_\infty}{(-q^4;q^4)_\infty}\sum_{n=0}^{\infty}i^{3n}(-q)^{3n^2+2n}(1+iq^{2n+1})\Bigg\}.\end{aligned} \tag{6.3.10}$$

First,

$$\begin{aligned}\mathrm{Re}\left\{(1+i)\sum_{n=0}^{\infty}(-i)^n(-q)^{n(n+1)/2}\right\}&=\sum_{n=0}^{\infty}q^{n(2n+1)}-\sum_{n=0}^{\infty}q^{(n+1)(2n+1)}\\&=\sum_{n=0}^{\infty}(-1)^nq^{n(n+1)/2}.\end{aligned} \tag{6.3.11}$$

Second,

$$\begin{aligned}\operatorname{Re}&\left\{\sum_{n=0}^{\infty} i^{3n+1}(-q)^{3n^2+2n}(1+iq^{2n+1})\right\}\\ &= -\sum_{n=0}^{\infty}(-1)^n q^{3(2n+1)^2+2(2n+1)} - \sum_{n=0}^{\infty}(-1)^n q^{3(2n)^2+4(2n)+1}\\ &= -q\sum_{n=0}^{\infty}(-1)^n q^{12n^2+8n}(1+q^{8n+4}).\end{aligned} \tag{6.3.12}$$

Substituting (6.3.11) and (6.3.12) into (6.3.10), we complete the proof after observing that

$$\frac{(-q;q^2)_\infty}{(-q^4;q^4)_\infty} = \frac{(q^2;q^4)_\infty(q^4;q^4)_\infty}{(q^8;q^8)_\infty(q;q^2)_\infty} = \frac{(q^2;q^2)_\infty}{(q^8;q^8)_\infty(q;q^2)_\infty} = \frac{\psi(q)}{(q^8;q^8)_\infty},$$

by (1.4.10). □

Entry 6.3.11 (p. 4). *For $a \neq 0$,*

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^n}{(-aq;q)_n(-q/a;q)_n} &= (1+a)\sum_{n=0}^{\infty}(-a)^n q^{n(n+1)/2}\\ &\quad - \frac{a(q;q^2)_\infty}{(-aq;q)_\infty(-q/a;q)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{2n} q^{n(n+1)}.\end{aligned}$$

Proof. For ease of notation, we shall prove the identity above with q replaced by q^2, in which case it may be rewritten as

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(q;q^2)_n(-q;q^2)_n q^{2n}}{(-aq^2;q^2)_n(-q^2/a;q^2)_n} &= (1+a)\sum_{n=0}^{\infty}(-a)^n q^{n(n+1)}\\ &\quad - \frac{a(q;q^2)_\infty(-q;q^2)_\infty}{(-aq^2;q^2)_\infty(-q^2/a;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{2n} q^{2n(n+1)}.\end{aligned}$$

Now in Theorem 6.2.1, replace q by q^2, a by $1/a$, and b by a. Then set $B = q$ and $A = a^{-1}q^{-1}$. Thus,

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(q;q^2)_n(-q;q^2)_n q^{2n}}{(-aq^2;q^2)_n(-q^2/a;q^2)_n}\\ &= -\frac{a(q;q^2)_\infty(-q;q^2)_\infty}{(-aq^2;q^2)_\infty(-q^2/a;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(aq;q^2)_m(aq)^m}{(-aq;q^2)_{m+1}}\\ &\quad + (1+a)\sum_{m=0}^{\infty}\frac{(-a;q^2)_{m+1}(-q^2;q^2)_m(-a)^m}{(-aq;q^2)_{m+1}(aq;q^2)_{m+1}}.\end{aligned} \tag{6.3.13}$$

Apply (1.7.1) with q replaced by q^2, and then with $\alpha = aq$, $\beta = -aq^3$, and $\tau = aq$. After dividing both sides of the resulting identity by $(1+aq)$, we find that

$$\sum_{m=0}^{\infty} \frac{(aq;q^2)_m (aq)^m}{(-aq;q^2)_{m+1}} = \sum_{n=0}^{\infty} (-1)^n a^{2n} q^{2n(n+1)}. \tag{6.3.14}$$

Next, invoke (1.2.1) with $h=2$, q replaced by q^2, and a, b, c, and t replaced by a^2q^2, $-a$, a^2q^2, and q^4, respectively, to deduce that

$$\begin{aligned}
S &:= \sum_{m=0}^{\infty} \frac{(-a;q^2)_{m+1}(-q^2;q^2)_m(-a)^m}{(-aq;q^2)_{m+1}(aq;q^2)_{m+1}} \\
&= \frac{1+a}{1-a^2q^2} \sum_{m=0}^{\infty} \frac{(-aq^2;q^2)_m(q^4;q^4)_m(-a)^m}{(q^2;q^2)_m(a^2q^6;q^4)_m} \\
&= \frac{(1+a)(a^2q^2;q^2)_\infty(q^4;q^4)_\infty}{(1-a^2q^2)(-a;q^2)_\infty(a^2q^6;q^4)_\infty} \sum_{n=0}^{\infty} \frac{(a^2q^2;q^4)_n(-a;q^2)_{2n}q^{4n}}{(q^4;q^4)_n(a^2q^2;q^2)_{2n}} \\
&= (aq^2;q^2)_\infty(q^4;q^4)_\infty \sum_{n=0}^{\infty} \frac{(-a;q^4)_n(-aq^2;q^4)_nq^{4n}}{(q^4;q^4)_n(a^2q^4;q^4)_n}.
\end{aligned} \tag{6.3.15}$$

To the far right side of (6.3.15), we apply (1.2.1) with q replaced by q^4, $h=1$, and a replaced by $-a$. After this, we set $b=-aq^2$, $c=a^2q^4$, and $t=q^4$ to finally arrive at

$$\begin{aligned}
S &= (aq^2;q^2)_\infty(q^4;q^4)_\infty \frac{(-aq^2;q^4)_\infty(-aq^4;q^4)_\infty}{(a^2q^4;q^4)_\infty(q^4;q^4)_\infty} \sum_{n=0}^{\infty} \frac{(-aq^2;q^4)_n(-aq^2)^n}{(-aq^4;q^4)_n} \\
&= \sum_{n=0}^{\infty} a^{2n}q^{4n^2+2n}(1-aq^{4n+2}) \\
&= \sum_{n=0}^{\infty} (-a)^n q^{n(n+1)},
\end{aligned} \tag{6.3.16}$$

where in the penultimate line we applied (1.7.1) with q replaced by q^4, α by $-aq^2$, β by $-aq^4$, and τ by $-aq^2$.

Substituting (6.3.14) and (6.3.16) into (6.3.13), we complete the proof of Entry 6.3.11. □

Entry 6.3.12 (p. 1). *If a, b, and $c \neq 0$ are complex, then*

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-aq)_n(-bq)_nq^{n+1}}{(-cq)_n} &= \sum_{n=1}^{\infty} \frac{(-1/c)_n(ab/c)^{n-1}q^{n(n+1)/2}}{(aq/c)_n(bq/c)_n} \\
&\quad - \frac{(-aq)_\infty(-bq)_\infty}{c(-cq)_\infty} \sum_{n=1}^{\infty} \frac{(ab/c^2)^{n-1}q^{n^2}}{(aq/c)_n(bq/c)_n}.
\end{aligned}$$

Proof. In Theorem 6.2.1, replace B by $-Bq$ and A by A/b. Then let $b \to 0$ to deduce that

$$\sum_{n=0}^{\infty} \frac{(-Bq)_n(-Aq)_n q^n}{(-aq)_n} = -\frac{(-Bq)_\infty(-Aq)_\infty}{a(-aq)_\infty} \sum_{m=0}^{\infty} \frac{(Aq/a)^m}{(Bq/a)_{m+1}} + \sum_{m=0}^{\infty} \frac{(-1/a)_{m+1}(AB/a)^m q^{m(m+3)/2}}{(Bq/a)_{m+1}(Aq/a)_{m+1}}. \tag{6.3.17}$$

Next, in (2.1.1), replace a, b, c, d, and e by q, $1/\tau$, $1/\tau$, Bq^2/a, and Aq^2/a, respectively, and then let $\tau \to 0$. After multiplying both sides by $(1 - Bq/a)^{-1}(1 - Aq/a)^{-1}$ and simplifying, we find that

$$\sum_{n=0}^{\infty} \frac{(AB/a^2)^n q^{n^2+2n}}{(Bq/a)_{n+1}(Aq/a)_{n+1}} = \sum_{n=0}^{\infty} \frac{(Aq/a)^n}{(Bq/a)_{n+1}}. \tag{6.3.18}$$

Therefore, we may substitute the left-hand side of (6.3.18) into the first expression on the right-hand side of (6.3.17). After multiplying both sides of the resulting identity by q, we arrive at

$$\sum_{n=0}^{\infty} \frac{(-Bq)_n(-Aq)_n q^{n+1}}{(-aq)_n} = -\frac{(-Bq)_\infty(-Aq)_\infty}{a(-aq)_\infty} \sum_{n=1}^{\infty} \frac{(AB/a^2)^{n-1} q^{n^2}}{(Bq/a)_n(Aq/a)_n} + \sum_{n=1}^{\infty} \frac{(-1/a)_n(AB/a)^{n-1} q^{n(n+1)/2}}{(Bq/a)_n(Aq/a)_n},$$

and this is the desired result upon replacing A, B, and a by a, b, and c, respectively. ∎

Entry 6.3.13 (p. 30). *If $a \neq 0$, then*

$$(1+a)\sum_{n=0}^{\infty} \frac{(-1)^n (cq)_n a^{-n-1} b^n q^{n(n+1)/2}}{(-cq/a)_{n+1}(-bq)_n} = \sum_{n=0}^{\infty} \frac{(cq)_n q^n}{(-aq)_n(-bq)_n} + \frac{(cq)_\infty}{(-aq)_\infty(-bq)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n a^{-n-1} b^n q^{n(n+1)/2}}{(-cq/a)_{n+1}}.$$

Proof. In Theorem 6.2.1, set $A = 0$ and $B = cq$. Thus,

$$\sum_{n=0}^{\infty} \frac{(cq)_n q^n}{(-aq)_n(-bq)_n} = -\frac{(cq)_\infty}{(-aq)_\infty(-bq)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n a^{-n-1} b^n q^{n(n+1)/2}}{(-cq/a)_{n+1}} + (1+b)\sum_{n=0}^{\infty} \frac{(-1/a)_{n+1}(-b)^n}{(-cq/a)_{n+1}}.$$

To conclude the proof, we must show that

$$(1+b)\sum_{n=0}^{\infty}\frac{(-1/a)_{n+1}(-b)^n}{(-cq/a)_{n+1}}=(1+a)\sum_{n=0}^{\infty}\frac{(-1)^n(cq)_n a^{-n-1}b^n q^{n(n+1)/2}}{(-cq/a)_{n+1}(-bq)_n},$$

which follows from (2.1.3), after replacing a, b, d, and e by q, cq, $-cq^2/a$, and $-bq$, respectively, and then multiplying both sides by $(1+1/a)/(1+cq/a)$. □

Entry 6.3.14 (p. 3). *If $a \neq 0$, then*

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n q^{n^2}}{(-aq^2;q^2)_n(-q^2/a;q^2)_n}=(1+a)\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^{2n+1}}{(-aq;q)_{2n+1}}
+\frac{(q;q^2)_\infty}{(-aq;q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2}}{(-q^2/a;q^2)_n}.$$

Proof. In Entry 6.3.13, replace q by q^2, and then replace a, b, and c by aq, a, and $1/q$, respectively. Multiplying both sides of the resulting identity by $q(1+a)/(1+aq)$, we complete the proof. □

Entry 6.3.15 (p. 29). *We have*

$$\sum_{n=0}^{\infty}\frac{(q;q)_{2n}q^{2n}}{(q^6;q^6)_n}=\frac{3}{2}\sum_{n=0}^{\infty}q^{18n^2+3n}(1-q^{12n+3}+q^{18n+6}-q^{30n+15})
-\frac{1}{2}\frac{(q;q)_\infty}{(q^6;q^6)_\infty}\sum_{n=0}^{\infty}(-1)^n q^{3n^2+2n}(1+q^{2n+1}).$$

Proof. In Entry 6.3.13, replace q by q^2. Then set $c=1/q$, $a=-\omega$, and $b=-\omega^2$, where $\omega=e^{2\pi i/3}$. Consequently, after simplification and rearrangement, we find that

$$\sum_{n=0}^{\infty}\frac{(q;q)_{2n}q^{2n}}{(q^6;q^6)_n}=\frac{(q;q)_\infty}{\omega(q^6;q^6)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n\omega^n q^{n(n+1)}}{(q/\omega;q^2)_{n+1}}
-\omega^{-1}(1-\omega)\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n\omega^n q^{n^2+n}}{(q/\omega;q^2)_{n+1}(\omega^2q^2;q^2)_n}.$$

In order to conclude the proof of Entry 6.3.15, we may assume that q is real and we must show that

$$\operatorname{Re}\left(\omega^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n\omega^n q^{n(n+1)}}{(q/\omega;q^2)_{n+1}}\right)=-\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{3n^2+2n}(1+q^{2n+1}) \tag{6.3.19}$$

and

$$\operatorname{Re}\left(-\omega^{-1}(1-\omega)\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n\omega^n q^{n(n+1)}}{(\omega^2q;q^2)_{n+1}(\omega^2q^2;q^2)_n}\right)
=\frac{3}{2}\sum_{n=0}^{\infty}q^{18n^2+3n}\left(1-q^{12n+3}+q^{18n+6}-q^{30n+15}\right). \tag{6.3.20}$$

First, by Entry 9.5.1 of [31] with $a = -\omega^{-1}$, we see that

$$\omega^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n\omega^n q^{n(n+1)}}{(q/\omega;q^2)_{n+1}} = \omega^{-1}\sum_{n=0}^{\infty}(-1)^n q^{3n^2+2n}(1+\omega^{-1}q^{2n+1}).$$

Hence, assuming that q is real, we find that

$$\begin{aligned}\operatorname{Re}\left(\omega^{-1}\sum_{n=0}^{\infty}\frac{(-1)^n\omega^n q^{n(n+1)}}{(q/\omega;q^2)_{n+1}}\right) &= -\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{3n^2+2n} - \frac{1}{2}\sum_{n=0}^{\infty} q^{3n^2+4n+1}\\ &= -\frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{3n^2+2n}(1+q^{2n+1}),\end{aligned}$$

which establishes (6.3.19).

Secondly, for (6.3.20), we apply (2.1.3) with q replaced by q^2, and then with a, b, d, and e replaced by q^2, q, ω^2q^2, and ω^2q^3, respectively. Accordingly, we find that

$$\begin{aligned}&\operatorname{Re}\left(-\frac{\omega^{-1}(1-\omega)}{1-\omega^2q}\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n\omega^n q^{n(n+1)}}{(\omega^2q^3;q^2)_n(\omega^2q^2;q^2)_n}\right)\\ &\qquad\qquad = \operatorname{Re}\left((1-\omega^{-1})\sum_{n=0}^{\infty}\frac{(\omega^2q;q^2)_n(\omega^2q)^n}{(\omega^2q^2;q^2)_n}\right).\end{aligned}$$

Next, we apply (1.7.1) to the right-hand side above with q replaced by q^2, and then with $\alpha = \omega^2q$, $\beta = \omega^2q^2$, and $\tau = \omega^2q$. Thus,

$$\begin{aligned}&\operatorname{Re}\left(-\frac{\omega^{-1}(1-\omega)}{1-\omega^2q}\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n\omega^n q^{n(n+1)}}{(\omega^2q^3;q^2)_n(\omega^2q^2;q^2)_n}\right)\\ &= \operatorname{Re}\left((1-\omega^{-1})\sum_{n=0}^{\infty}\omega^n q^{2n^2+n}(1+\omega^2q^{2n+1})\right)\\ &= \frac{3}{2}\sum_{\substack{n=0\\ n\equiv 0 \pmod 3}}^{\infty} q^{2n^2+n} - \frac{3}{2}\sum_{\substack{n=0\\ n\equiv 1 \pmod 3}}^{\infty} q^{2n^2+n}\\ &\quad + \frac{3}{2}\sum_{\substack{n=0\\ n\equiv 1 \pmod 3}}^{\infty} q^{2n^2+3n+1} - \frac{3}{2}\sum_{\substack{n=0\\ n\equiv 2 \pmod 3}}^{\infty} q^{2n^2+3n+1}\\ &= \frac{3}{2}\sum_{n=0}^{\infty} q^{18n^2+3n}\left(1-q^{12n+3}+q^{18n+6}-q^{30n+15}\right).\end{aligned}$$

Hence, we have demonstrated the truth of (6.3.20), and with it the validity of Entry 6.3.15. ∎

The final entry in this section is probably the most speculative in this chapter, because at the bottom of page 4 in his lost notebook, Ramanujan wrote down only the left-hand side of the following result; no right-hand side was given by Ramanujan. We think that our interpretation of Ramanujan's intention is accurate, because it too is a special case of Theorem 6.2.1. However, in light of the fact that no partial theta series appears, Ramanujan may have decided not to write down the complete result.

Entry 6.3.16 (p. 4). *We have*

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^4)_n q^{2n+1}}{(-q;q)_{2n+1}} = 2q \sum_{n=0}^{\infty} \frac{(-1)^n (q;-q)_{2n} q^n}{(-q^2;q^4)_{n+1}} \\ - \frac{q(-q^2;q^4)_\infty}{(-q;q)_\infty} \sum_{n=0}^{\infty} \frac{(-q^4;q^4)_n q^{2n^2+2n}(1-q^{4n+3})}{(-q^2;q^4)_{n+1}}.$$

Proof. In Theorem 6.2.1, replace q by q^2 and then set $A = -iq^{-2}$, $B = -iq$, $a = 1$, and $b = q$. Finally, multiply both sides by $q/(1+q)$. After simplification, we find that

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^4)_n q^{2n+1}}{(-q;q)_{2n+1}} = -\frac{q(-q^2;q^4)_\infty}{(-q;q)_\infty} \sum_{n=0}^{\infty} \frac{(iq^2;q^2)_n (-iq)^n}{(iq;q^2)_{n+1}} \\ + 2q \sum_{n=0}^{\infty} \frac{(-1)^n (q;-q)_{2n} q^n}{(-q^2;q^4)_{n+1}}.$$

To conclude our proof, we must show that

$$\sum_{n=0}^{\infty} \frac{(iq^2;q^2)_n (-iq)^n}{(iq;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-q^4;q^4)_n q^{2n^2+2n}(1-q^{4n+3})}{(-q^2;q^4)_{n+1}}.$$

However, this last result follows from (1.7.1) by replacing q by q^2, then setting $\alpha = iq^2$, $\beta = iq^3$, and $\tau = -iq$, and lastly dividing both sides by $(1-iq)$. □

6.4 The function $\psi(a, q)$

In this section, we collect together some results, mostly from page 38 of the lost notebook, connected with the partial theta function

$$\psi(a,q) := \sum_{n=0}^{\infty} a^n q^{n(n+1)/2}.$$

Note that $\psi(1,q) = \psi(q)$. In some ways, these results fit reasonably and naturally into several other locations in our development. However, they seem most appealing when exhibited independently.

Entry 6.4.1 (p. 38). *If* $\omega = e^{2\pi i/3}$, *then*

$$\psi(a,q) + \omega\psi(a,\omega q) + \omega^2\psi(a,\omega^2 q) = 0.$$

Proof. The proposed identity easily follows from the facts that there are no powers of q congruent to 2 modulo 3 and that for each index of summation n modulo 3, the sum of the nth terms of the three series is equal to 0, because the sum of the three cube roots of unity equals 0. □

Entry 6.4.2 (p. 38). *If* $\omega = e^{2\pi i/3}$, *then*

$$\psi(a,q) + \omega^2\psi(a,\omega q) + \omega\psi(a,\omega^2 q) = 3aq\psi(a^3,q^9).$$

Proof. We have

$$\begin{aligned}\psi(a,q) + \omega^2\psi(a,\omega q) + \omega\psi(a,\omega^2 q) &= 3\sum_{\substack{n=0\\ n(n+1)/2\equiv 1 \pmod 3}}^{\infty} a^n q^{n(n+1)/2}\\ &= 3\sum_{n=0}^{\infty} a^{3n+1}q^{(3n+1)(3n+2)/2}\\ &= 3aq\sum_{n=0}^{\infty} a^{3n}q^{9n(n+1)/2} = 3aq\psi(a^3,q^9).\end{aligned}$$

Thus, the proof is complete. □

Entry 6.4.3 (p. 38). *For any complex number* a,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}q^{n(3n+1)/2}}{(-aq;q^3)_{n+1}} &= \frac{1}{2}\sum_{n=0}^{\infty} a^{3n}q^{3n(3n+1)/2}(1-a^2q^{6n+3})\\ &\quad + \frac{1}{2}\left\{\psi(a,q) - 3aq\psi(a^3,q^9)\right\}.\end{aligned}$$

Proof. In Entry 9.5.1 of [31], replace q by $q^{3/2}$ and a by $aq^{-1/2}$ to deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}q^{n(3n+1)/2}}{(-aq;q^3)_{n+1}} &= \sum_{n=0}^{\infty} a^{3n}q^{3n(3n+1)/2}(1-aq^{3n+1}) \qquad (6.4.1)\\ &= \sum_{\substack{n=0\\ n\equiv 0 \pmod 3}}^{\infty} a^n q^{n(n+1)/2} - \sum_{\substack{n=0\\ n\equiv 1 \pmod 3}}^{\infty} a^n q^{n(n+1)/2}.\end{aligned}$$

On the other hand,

$$\frac{1}{2}\sum_{n=0}^{\infty} a^{3n}q^{3n(3n+1)/2}(1-a^2q^{6n+3}) + \frac{1}{2}\left\{\psi(a,q) - 3aq\psi(a^3,q^9)\right\}$$

$$
\begin{aligned}
&= \frac{1}{2} \sum_{\substack{n=0 \\ n\equiv 0 \pmod 3}}^{\infty} a^n q^{n(n+1)/2} - \frac{1}{2} \sum_{\substack{n=0 \\ n\equiv 2 \pmod 3}}^{\infty} a^n q^{n(n+1)/2} \\
&\quad + \frac{1}{2} \sum_{n=0}^{\infty} a^n q^{n(n+1)/2} - \frac{3}{2} \sum_{\substack{n=1 \\ n\equiv 1 \pmod 3}}^{\infty} a^n q^{n(n+1)/2} \\
&= \sum_{\substack{n=0 \\ n\equiv 0 \pmod 3}}^{\infty} a^n q^{n(n+1)/2} - \sum_{\substack{n=0 \\ n\equiv 1 \pmod 3}}^{\infty} a^n q^{n(n+1)/2}. \qquad (6.4.2)
\end{aligned}
$$

Comparing (6.4.2) and (6.4.1), we see that Entry 6.4.3 has been established. □

Entry 6.4.4 (p. 38). *For each complex number a,*

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1} q^{(n+1)(3n+1)/2}}{(-aq^2;q^3)_{n+1}} &= \frac{1}{2} \sum_{n=0}^{\infty} a^{3n} q^{3n(3n+1)/2} (1 - a^2 q^{6n+3}) \\
&\quad - \frac{1}{2} \left\{ \psi(a,q) - 3aq\psi(a^3,q^9) \right\}.
\end{aligned}
$$

Proof. Replacing a by aq in (6.4.1) and multiplying both sides by aq, we see that

$$
\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1} q^{(n+1)(3n+1)/2}}{(-aq^2;q^3)_{n+1}} &= \sum_{n=0}^{\infty} a^{3n+1} q^{(3n+1)(3n+2)/2} (1 - aq^{3n+2}) \\
&= \sum_{\substack{n=0 \\ n\equiv 1 \pmod 3}}^{\infty} a^n q^{n(n+1)/2} - \sum_{\substack{n=0 \\ n\equiv 2 \pmod 3}}^{\infty} a^n q^{n(n+1)/2}. \qquad (6.4.3)
\end{aligned}
$$

On the other hand, following the argument used in the proof of Entry 6.4.3, we find that

$$
\begin{aligned}
&\frac{1}{2} \sum_{n=0}^{\infty} a^{3n} q^{3n(3n+1)/2} (1 - a^2 q^{6n+3}) \quad - \frac{1}{2} \left\{ \psi(a,q) - 3aq\psi(a^3,q^9) \right\} \\
&= \frac{1}{2} \sum_{\substack{n=0 \\ n\equiv 0 \pmod 3}}^{\infty} a^n q^{n(n+1)/2} - \frac{1}{2} \sum_{\substack{n=0 \\ n\equiv 2 \pmod 3}}^{\infty} a^n q^{n(n+1)/2} \\
&\quad - \frac{1}{2} \sum_{n=0}^{\infty} a^n q^{n(n+1)/2} + \frac{3}{2} \sum_{\substack{n=0 \\ n\equiv 1 \pmod 3}}^{\infty} a^n q^{n(n+1)/2} \\
&= \sum_{\substack{n=0 \\ n\equiv 1 \pmod 3}}^{\infty} a^n q^{n(n+1)/2} - \sum_{\substack{n=0 \\ n\equiv 2 \pmod 3}}^{\infty} a^n q^{n(n+1)/2}. \qquad (6.4.4)
\end{aligned}
$$

Substituting (6.4.4) into (6.4.3), we complete the proof of Entry 6.4.4. □

Entry 6.4.5 (p. 39). *For any complex number a,*

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(3n+1)/2}}{(-aq;q^3)_{n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1} q^{(n+1)(3n+2)/2}}{(-aq^2;q^3)_{n+1}} = \sum_{n=0}^{\infty} a^{3n} q^{3n(3n+1)/2}(1-a^2q^{6n+3}).$$

Proof. Add together Entries 6.4.3 and 6.4.4. □

Entry 6.4.6 (p. 4). *For $a \neq 0$,*

$$\left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty} \frac{(-1)^n(-aq;q)_n(-q/a;q)_n}{(q;q^2)_{n+1}} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(a^n+a^{-n-1})q^{n(n+1)/2}. \tag{6.4.5}$$

The series on the left-hand side above is not convergent. Thus, we need to introduce a certain type of summability to give meaning to (6.4.5). We shall prove an identity for

$$\left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty} \frac{(-aq;q)_n(-q/a;q)_n(-t)^n}{(q;q^2)_{n+1}},$$

and when we let $t \to 1^-$, the right-hand side will be seen to converge to the right-hand side of (6.4.5).

Proof. After applying (1.7.3) below, we replace the index j by $N+n+1$. Thus, after some initial rearrangement, we find that

$$\begin{aligned}
&\left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty} \frac{(-aq;q)_n(-q/a;q)_n(-t)^n}{(q;q^2)_{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(-t)^n a^{-n-1} q^{n(n+1)/2}(-aq^{-n};q)_{2n+1}}{(q;q^2)_{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(-t)^n a^{-n-1} q^{n(n+1)/2}}{(q;q^2)_{n+1}} \sum_{j=-\infty}^{\infty} \begin{bmatrix} 2n+1 \\ j \end{bmatrix} a^j q^{j(j-1)/2-nj} \\
&= \sum_{N=-\infty}^{\infty} a^N \sum_{n=0}^{\infty} \frac{(-t)^n q^{n(n+1)/2}}{(q;q^2)_{n+1}} \begin{bmatrix} 2n+1 \\ N+n+1 \end{bmatrix} q^{(N+n+1)(N+n)/2-n(N+n+1)} \\
&= \sum_{N=-\infty}^{\infty} a^N q^{N(N+1)/2} \sum_{n=0}^{\infty} \frac{(q^2;q^2)_n(-t)^n}{(q;q)_{n-N}(q;q)_{n+N+1}} \\
&=: \sum_{N=-\infty}^{\infty} a^N q^{N(N+1)/2} C_N(t),
\end{aligned} \tag{6.4.6}$$

where we note that, for $N \geq 0$,

$$C_N(t) = C_{-N-1}(t). \tag{6.4.7}$$

Hence, we shall now evaluate $C_N(t)$ for $N \geq 0$.

To that end, we first replace n by $n+N$ and then we apply (1.2.9) with a, b, c, and t replaced by $-q^{N+1}$, q^{N+1}, q^{2N+2}, and $-t$, respectively, to deduce that

$$\begin{aligned}
C_N(t) &= \sum_{n=0}^{\infty} \frac{(q^2;q^2)_{n+N}(-t)^{n+N}}{(q;q)_n(q;q)_{n+2N+1}} \\
&= \frac{(q^2;q^2)_N(-t)^N}{(q;q)_{2N+1}} \sum_{n=0}^{\infty} \frac{(q^{N+1};q)_n(-q^{N+1};q)_n(-t)^n}{(q;q)_n(q^{2N+2};q)_n} \\
&= \frac{(q^2;q^2)_N(-t)^N}{(q;q)_{2N+1}} \frac{(q^{N+1};q)_\infty(-tq^{N+1};q)_\infty}{(q^{2N+2};q)_\infty(-t;q)_\infty} \\
&\quad \times \sum_{n=0}^{\infty} \frac{(t;q)_n(q^{N+1};q)_n q^{n(N+1)}}{(q;q)_n(-tq^{N+1};q)_n} \\
&= \frac{(-q;q)_N(-tq^{N+1};q)_\infty(-t)^N}{(-t;q)_\infty} \sum_{n=0}^{\infty} \frac{(t;q)_n(q^{N+1};q)_n q^{n(N+1)}}{(q;q)_n(-tq^{N+1};q)_n}.
\end{aligned}$$

Hence, we deduce immediately that

$$\lim_{t\to 1^-} C_N(t) = \frac{1}{2}(-1)^N. \tag{6.4.8}$$

Also, using (6.4.7) and (6.4.8), we deduce that

$$\lim_{t\to 1^-} C_{-N}(t) = \lim_{t\to 1^-} C_{N-1}(t) = -\frac{1}{2}(-1)^N. \tag{6.4.9}$$

Letting $t \to 1^-$ in (6.4.6) and using (6.4.8) and (6.4.9), we obtain the right-hand side of (6.4.5), as we promised immediately after the statement of Entry 6.4.6. $\square$

We note that Entry 6.4.6 is identical to Entry 5.4.3 in Chapter 5, where an entirely different method was used.

6.5 Euler's Identity and Its Extensions

The identity of Euler to which we refer is [18, p. 19, equation (2.25)]

$$\sum_{n=0}^{\infty} \frac{q^n}{(q)_n} = \frac{1}{(q)_\infty}. \tag{6.5.1}$$

In this section we prove a number of identities from the lost notebook that are slight variations on (6.5.1) and fit naturally into this chapter. The remainder of the results in this section are so closely related in form that it is appropriate that they should also appear in this section.

In the first entry and in others as well, we record a pair of identities, because their proofs are inextricably intertwined.

Entry 6.5.1 (p. 31). *We have*

$$\sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_{2n}} = \sum_{n=0}^{\infty} q^{12n^2+n}(1-q^{22n+11}) + q\sum_{n=0}^{\infty} q^{12n^2+7n}(1-q^{10n+5}) \tag{6.5.2}$$

and

$$\sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} q^{12n^2+5n}(1-q^{14n+7}) + q^2\sum_{n=0}^{\infty} q^{12n^2+11n}(1-q^{2n+1}). \tag{6.5.3}$$

First Proof of Entry 6.5.1. We put $a = i$ in Entry 6.3.2, assume that q is real, and take the real parts of both sides to arrive at

$$\begin{aligned} h(q) := \sum_{n=0}^{\infty} \frac{q^n}{(-q^2;q^2)_n} &= \operatorname{Re}\left\{(1+i)\sum_{n=0}^{\infty} i^{3n} q^{n(3n+1)/2}(1+q^{2n+1})\right\} \\ &= \sum_{n=0}^{\infty}(-1)^n q^{n(6n+1)}(1+q^{4n+1}) + \sum_{n=0}^{\infty}(-1)^n q^{(2n+1)(3n+2)}(1+q^{4n+3}). \end{aligned}$$

Let $f(q)$ denote the left-hand side of (6.5.2). Then

$$\begin{aligned} f(q^2) &= \sum_{n=0}^{\infty} \frac{q^{2n}}{(-q^2;q^2)_{2n}} \\ &= \frac{1}{2}\sum_{n=0}^{\infty} \frac{(1+(-1)^n)q^n}{(-q^2;q^2)_n} \\ &= \frac{1}{2}\left(h(q)+h(-q)\right) \\ &= \text{Even part of}\left\{\sum_{n=0}^{\infty}(-1)^n q^{n(6n+1)}(1+q^{4n+1}) \right. \\ &\qquad \left. + \sum_{n=0}^{\infty}(-1)^n q^{(2n+1)(3n+2)}(1+q^{4n+3})\right\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} q^{24n^2+2n} - \sum_{n=0}^{\infty} q^{24n^2+34n+12} \\
&\quad + \sum_{n=0}^{\infty} q^{24n^2+14n+2} - \sum_{n=0}^{\infty} q^{24n^2+46n+22}.
\end{aligned}
$$

Replacing q^2 by q in the extremal sides of this last equality yields (6.5.2).

Let $g(q)$ denote the left-hand side of (6.5.3). Then

$$
\begin{aligned}
qg(q^2) &= \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(-q^2;q^2)_{2n+1}} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(1-(-1)^n)q^n}{(-q^2;q^2)_n} \\
&= \frac{1}{2}\left(h(q) - h(-q)\right) \\
&= \text{Odd part of} \left\{ \sum_{n=0}^{\infty} (-1)^n q^{n(6n+1)}(1+q^{4n+1}) \right. \\
&\qquad \left. - \sum_{n=0}^{\infty} (-1)^n q^{(2n+1)(3n+2)}(1+q^{4n+3}) \right\} \\
&= -\sum_{n=0}^{\infty} q^{24n^2+26n+7} + \sum_{n=0}^{\infty} q^{24n^2+10n+1} \\
&\quad - \sum_{n=0}^{\infty} q^{24n^2+38n+15} + \sum_{n=0}^{\infty} q^{24n^2+22n+5}.
\end{aligned}
$$

Dividing the last set of equalities by q and replacing q^2 by q, we complete the proof of (6.5.3). $\square$

Second Proof of Entry 6.5.1. We prove (6.5.2), as (6.5.3) can be similarly proved.

Replacing q by q^2 in (6.5.2), we obtain the identity

$$
\sum_{n=0}^{\infty} \frac{q^{2n}}{(-q^2;q^2)_{2n}} = \sum_{n=0}^{\infty} q^{24n^2+2n}(1-q^{44n+22}) + q^2 \sum_{n=0}^{\infty} q^{24n^2+14n}(1-q^{20n+10}). \tag{6.5.4}
$$

The left-hand side generates partitions $\lambda = \lambda_1, \ldots, \lambda_r$ into an even number of odd parts with weight $(-1)^{(\lambda_1-1)/2}$. Clearly, λ is a partition of an even number. By Sylvester's map [26], such partitions have a one-to-one correspondence with partitions into distinct parts. For a partition λ of an even number $2N$ into odd parts, let $\mu = \mu_1, \ldots, \mu_s$ be the image of λ under Sylvester's bijection, which is a partition of $2N$ into distinct parts. Note that Sylvester's bijection preserves the following statistic:

$$\ell(\lambda) + \frac{\lambda_1 - 1}{2} = \mu_1,$$

where $\ell(\lambda)$ denotes the number of parts in the partition λ. Since $\ell(\lambda)$ is even, we see that

$$(-1)^{(\lambda_1-1)/2} = (-1)^{\mu_1}.$$

Thus it follows that

$$\sum_{n=0}^{\infty} \frac{q^{2n}}{(-q^2;q^2)_{2n}} = \sum_{N=0}^{\infty} \sum_{\lambda \in O_{2N}} (-1)^{(\lambda_1-1)/2} q^{2N} = \sum_{N=0}^{\infty} \sum_{\mu \in D_{2N}} (-1)^{\mu_1} q^{2N}, \tag{6.5.5}$$

where O_{2N} and D_{2N} are the sets of partitions of $2N$ into odd parts and distinct parts, respectively. We now apply the involution for Euler's pentagonal number theorem [18, pp. 10–12], in which we compare the smallest part and the number of consecutive parts including the largest part. Note that in the pentagonal number theorem, partitions π have weight $(-1)^{\ell(\pi)}$. However, the involutive proof still works for our setting, since we move the smallest part to the right of the consecutive parts or subtract 1 from each of the consecutive parts in order to add the number of the consecutive parts as a new part. Thus only the partitions of the even pentagonal numbers survive under the involution in our setting, too. It is easy to see that

$$\begin{aligned} n(3n+1)/2 &\equiv 0 \pmod 2), \quad \text{if } n \equiv 0, 1 \pmod 4, \\ n(3n-1)/2 &\equiv 0 \pmod 2), \quad \text{if } n \equiv 0, 3 \pmod 4. \end{aligned}$$

When $n \equiv 0, 1 \pmod 4$, the surviving partition of $n(3n+1)/2$ has parts $2n, 2n-1, \ldots, n+1$. The largest part of the partition is even. When $n \equiv 0, 3 \pmod 4$, the largest part of the partition of $n(3n-1)/2$ is odd. Then,

$$\begin{aligned} &\sum_{N=0}^{\infty} \sum_{\mu \in D_{2N}} (-1)^{\mu_1} q^{2N} \qquad (6.5.6) \\ &= \sum_{\substack{n=0 \\ n(3n+1)/2 \equiv 0 \pmod 2}}^{\infty} q^{n(3n+1)/2} - \sum_{\substack{n=1 \\ n(3n-1)/2 \equiv 0 \pmod 2}}^{\infty} q^{n(3n-1)/2} \\ &= \sum_{n=0}^{\infty} q^{24n^2+2n}(1-q^{44n+22}) + q^2 \sum_{n=0}^{\infty} q^{24n^2+14n}(1-q^{20n+10}). \end{aligned}$$

Hence, by (6.5.5) and (6.5.6), we complete the proof of (6.5.4), and hence also of (6.5.2). □

The second proof of Entry 6.5.1 is taken from a paper by Berndt, Kim, and Yee [73]. Likewise, a bijective proof of Entry 6.5.2 can also be found in the same paper. Another proof of Entry 6.5.2 can be found in a paper by W. Chu and C. Wang [129].

Entry 6.5.2 (p. 31). *Recall that $\varphi(q)$, $f(-q)$, and $f(a,b)$ are defined by* (1.4.9), (1.4.11), *and* (1.4.8), *respectively. Then*

$$\sum_{n=0}^{\infty} \frac{q^n}{(q)_{2n}} = \frac{1}{\varphi(-q)} \left(\sum_{n=-\infty}^{\infty} q^{12n^2+n} - q \sum_{n=-\infty}^{\infty} q^{12n^2+7n} \right) \tag{6.5.7}$$

$$= \frac{f(q^5, q^3)}{f(-q)} \tag{6.5.8}$$

and

$$\sum_{n=0}^{\infty} \frac{q^n}{(q)_{2n+1}} = \frac{1}{\varphi(-q)} \left(\sum_{n=-\infty}^{\infty} q^{12n^2+5n} - q^2 \sum_{n=-\infty}^{\infty} q^{12n^2+11n} \right) \tag{6.5.9}$$

$$= \frac{f(q^7, q)}{f(-q)}. \tag{6.5.10}$$

These four identities are included in this one entry, because their proofs are intertwined. L. Carlitz [98] represented each of the two series on the left-hand sides as infinite products, from which it is easy to prove each of the four assertions. However, we proceed from scratch.

Proof. This proof completely parallels the proof of Entry 6.5.1. However, the key result at the beginning of the proof is more elementary in the proof at hand.

Let

$$j(q) := \sum_{n=0}^{\infty} \frac{q^n}{(q^2;q^2)_n} = \frac{1}{(q;q^2)_\infty}, \tag{6.5.11}$$

by (1.2.4). Let $c(q)$ denote the left-hand side of (6.5.7). Then, by (6.5.11) and by two applications of the Jacobi triple product identity (1.4.8),

$$\begin{aligned}
c(q^2) &= \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^2;q^2)_{2n}} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(1+(-1)^n)q^n}{(q^2;q^2)_n} \\
&= \frac{1}{2} (j(q) + j(-q)) \\
&= \frac{1}{2} \left(\frac{1}{(q;q^2)_\infty} + \frac{1}{(-q;q^2)_\infty} \right) \\
&= \frac{1}{2(q^2;q^2)_\infty} \big((-q;q^4)_\infty (-q^3;q^4)_\infty (q^4;q^4)_\infty \\
&\qquad + (q;q^4)_\infty (q^3;q^4)_\infty (q^4;q^4)_\infty \big) \\
&= \frac{1}{2(q^2;q^2)_\infty} \left(\sum_{n=-\infty}^{\infty} q^{2n^2+n} + \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2+n} \right)
\end{aligned}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{8n^2+2n} = \frac{f(q^{10},q^6)}{f(-q^2)},$$

which is (6.5.8) once q^2 is replaced throughout by q.

Now, by (1.4.8), Euler's identity, and the product representation for $\varphi(-q)$ in (1.4.9),

$$\begin{aligned}
\frac{f(q^5,q^3)}{f(-q)} &= \frac{(-q^3;q^8)_\infty(-q^5;q^8)_\infty(q^8;q^8)_\infty}{(q;q)_\infty} \\
&= \frac{(-q^3;q^8)_\infty(-q^5;q^8)_\infty(q^8;q^8)_\infty(q;q^2)_\infty}{\varphi(-q)} \\
&= \frac{(q;q^8)_\infty(q^7;q^8)_\infty(q^{10};q^{16})_\infty(q^6;q^{16})_\infty(q^8;q^8)_\infty}{\varphi(-q)} \\
&= \frac{1}{\varphi(-q)} \sum_{n=-\infty}^{\infty} q^{12n^2-n}(1-q^{8n+1}),
\end{aligned}$$

where in the last step we applied the quintuple product identity (3.1.2) with q replaced by q^8 and with $z=-q$. It is easily seen that this last identity is equivalent to (6.5.7).

Now let $d(q)$ denote the left-hand side of (6.5.9). Then, by (6.5.11) and by two applications of the Jacobi triple product identity (1.4.8),

$$\begin{aligned}
qd(q^2) &= \sum_{n=0}^{\infty} \frac{q^{2n+1}}{(q^2;q^2)_{2n+1}} \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(1-(-1)^n)q^n}{(q^2;q^2)_n} \\
&= \frac{1}{2}\left(j(q)-j(-q)\right) \\
&= \frac{1}{2}\left(\frac{1}{(q;q^2)_\infty} - \frac{1}{(-q;q^2)_\infty}\right) \\
&= \frac{1}{2(q^2;q^2)_\infty}\big((-q;q^4)_\infty(-q^3;q^4)_\infty(q^4;q^4)_\infty \\
&\qquad\qquad -(q;q^4)_\infty(q^3;q^4)_\infty(q^4;q^4)_\infty\big) \\
&= \frac{1}{2(q^2;q^2)_\infty}\left(\sum_{n=-\infty}^{\infty} q^{2n^2+n} - \sum_{n=-\infty}^{\infty} (-1)^n q^{2n^2+n}\right) \\
&= \frac{q}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} q^{8n^2-6n} = \frac{qf(q^2,q^{14})}{f(-q^2)}.
\end{aligned}$$

Divide by q throughout the display above and then replace q^2 by q to achieve the identity in (6.5.10).

Now, by the Jacobi triple product identity (1.4.8), the product representation for $\varphi(-q)$ given in (1.4.9), and Euler's identity,

$$\begin{aligned}
\frac{f(q,q^7)}{f(-q)} &= \frac{(-q;q^8)_\infty(-q^7;q^8)_\infty(q^8;q^8)_\infty}{(q;q)_\infty} \\
&= \frac{(-q;q^8)_\infty(-q^7;q^8)_\infty(q^8;q^8)_\infty(q;q^2)_\infty}{\varphi(-q)} \\
&= \frac{(q^3;q^8)_\infty(q^5;q^8)_\infty(q^8;q^8)_\infty(q^2;q^{16})_\infty(q^{14};q^{16})_\infty}{\varphi(-q)} \\
&= \frac{1}{\varphi(-q)}\sum_{n=-\infty}^{\infty} q^{12n^2+5n}(1-q^{8n+3}) \\
&= \frac{1}{\varphi(-q)}\left(\sum_{n=-\infty}^{\infty} q^{12n^2+5n} - q^2\sum_{n=-\infty}^{\infty} q^{12n^2+11n}\right),
\end{aligned}$$

where in the penultimate line we applied the quintuple product identity (3.1.2) with q replaced by q^8, and $z=-q^3$, and where in the last line we replaced n by $-n-1$ in the second sum. □

The final entry in this section requires some auxiliary results that are closely related in form to the series appearing in Entry 6.5.1.

Lemma 6.5.1. *For each nonnegative integer N and $d \neq 0, q^{-n}$, $1 \le n < \infty$,*

$$\sum_{n=0}^{\infty} \frac{q^{(N+1)n}}{(dq)_n} = \frac{(q)_N}{d^{N+1}(dq)_\infty} - \frac{(1-d)(q)_N}{d^{N+1}}\sum_{n=0}^{N}\frac{d^n}{(q)_n}.$$

Proof. Apply (1.2.9) with $a=0$, $b=q$, $c=dq$, and $t=q^{N+1}$ to find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{(N+1)n}}{(dq)_n} &= \frac{1-d}{1-q^{N+1}}\sum_{n=0}^{\infty}\frac{d^n}{(q^{N+2})_n} \\
&= \frac{(1-d)(q)_N}{d^{N+1}}\sum_{n=0}^{\infty}\frac{d^{n+N+1}}{(q)_{n+N+1}} \\
&= \frac{(1-d)(q)_N}{d^{N+1}}\left(\sum_{n=0}^{\infty}\frac{d^n}{(q)_n} - \sum_{n=0}^{N}\frac{d^n}{(q)_n}\right) \\
&= \frac{(q)_N}{d^{N+1}(dq)_\infty} - \frac{(1-d)(q)_N}{d^{N+1}}\sum_{n=0}^{N}\frac{d^n}{(q)_n},
\end{aligned}$$

where the last line follows from (1.2.2). From our application of (1.2.9), it is required that $0 < d < 1$, but this condition can be relaxed by appealing to analytic continuation. □

Lemma 6.5.2. *For each nonnegative integer m,*

$$(q;q^2)_m \sum_{n=0}^{2m} \frac{(-1)^n}{(q;q)_n} = \sum_{n=0}^{m} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_{m-n}}.$$

Proof. Denote the left- and right-hand sides of the lemma by L_m and R_m, respectively. Clearly, $L_0 = R_0 = 1$. We show that L_m and R_m satisfy the same two-term recurrence relation, and this suffices to prove the lemma.

First, for $m \geq 0$,

$$\begin{aligned} L_{m+1} &= (1-q^{2m+1})(q;q^2)_m \left(\sum_{n=0}^{2m} \frac{(-1)^n}{(q;q)_n} - \frac{1}{(q;q)_{2m+1}} + \frac{1}{(q;q)_{2m+2}} \right) \\ &= (1-q^{2m+1})L_m + (q;q^2)_m(1-q^{2m+1}) \frac{q^{2m+2}}{(q;q)_{2m+2}} \\ &= (1-q^{2m+1})L_m + \frac{q^{2m+2}}{(q^2;q^2)_{m+1}}. \end{aligned} \tag{6.5.12}$$

Second, for $m \geq 0$, splitting off the term for $n = 0$, we find that

$$\begin{aligned} R_{m+1} - R_m &= \sum_{n=0}^{m+1} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_{m+1-n}} \left(1 - (1 - q^{2m+2-2n})\right) \\ &= \frac{q^{2m+2}}{(q^2;q^2)_{m+1}} + q^{2m+1} \sum_{n=1}^{m+1} \frac{(-1)^n q^{(n-1)^2}}{(q^2;q^2)_{m+1-n}} \\ &= \frac{q^{2m+2}}{(q^2;q^2)_{m+1}} - q^{2m+1} R_m. \end{aligned} \tag{6.5.13}$$

Hence, from (6.5.12) and (6.5.13), we see that L_m and R_m satisfy the same recurrence relation. Thus, by mathematical induction, $L_m = R_m$ for all $m \geq 0$. ∎

Entry 6.5.3 (p. 21). *For any complex number a,*

$$\sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_n (aq;q^2)_n} = 2 \sum_{n=0}^{\infty} \frac{a^n q^{2n^2}}{(aq;q^2)_n} - \frac{(q;q^2)_\infty}{(aq;q^2)_\infty} \sum_{n=0}^{\infty} (-1)^n (q;q^2)_n a^n q^{n^2}.$$

Proof. By (1.2.6) and Lemma 6.5.1 with $N = 2m$ and $d = -1$,

$$\begin{aligned} S &:= \sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_n (aq;q^2)_n} \\ &= \frac{1}{(aq;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_n} (aq^{2n+1};q^2)_\infty \\ &= \frac{1}{(aq;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^n}{(-q;q)_n} \sum_{m=0}^{\infty} \frac{(-1)^m a^m q^{m^2+2nm}}{(q^2;q^2)_m} \end{aligned}$$

$$= \frac{1}{(aq;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m a^m q^{m^2}}{(q^2;q^2)_m} \sum_{n=0}^{\infty} \frac{q^{(2m+1)n}}{(-q;q)_n}$$

$$= \frac{1}{(aq;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-1)^m a^m q^{m^2}}{(q^2;q^2)_m} \left\{ -\frac{(q;q)_{2m}}{(-q;q)_\infty} + 2(q;q)_{2m} \sum_{n=0}^{2m} \frac{(-1)^n}{(q;q)_n} \right\}$$

$$= -\frac{(q;q^2)_\infty}{(aq;q^2)_\infty} \sum_{m=0}^{\infty} (-1)^m (q;q^2)_m a^m q^{m^2}$$

$$+ \frac{2}{(aq;q^2)_\infty} \sum_{m=0}^{\infty} (-1)^m (q;q^2)_m a^m q^{m^2} \sum_{n=0}^{2m} \frac{(-1)^n}{(q;q)_n},$$

by Euler's identity. Next, apply Lemma 6.5.2, replace m by $m+n$, invoke (1.2.6) again, and utilize (1.2.2) to find that

$$S = -\frac{(q;q^2)_\infty}{(aq;q^2)_\infty} \sum_{m=0}^{\infty} (-1)^m (q;q^2)_m a^m q^{m^2}$$

$$+ \frac{2}{(aq;q^2)_\infty} \sum_{m=0}^{\infty} (-1)^m a^m q^{m^2} \sum_{n=0}^{m} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_{m-n}}$$

$$= -\frac{(q;q^2)_\infty}{(aq;q^2)_\infty} \sum_{m=0}^{\infty} (-1)^m (q;q^2)_m a^m q^{m^2}$$

$$+ \frac{2}{(aq;q^2)_\infty} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^m a^{m+n} q^{m^2+2mn+2n^2}}{(q^2;q^2)_m}$$

$$= -\frac{(q;q^2)_\infty}{(aq;q^2)_\infty} \sum_{m=0}^{\infty} (-1)^m (q;q^2)_m a^m q^{m^2}$$

$$+ \frac{2}{(aq;q^2)_\infty} \sum_{n=0}^{\infty} a^n q^{2n^2} (aq^{2n+1};q^2)_\infty$$

$$= -\frac{(q;q^2)_\infty}{(aq;q^2)_\infty} \sum_{m=0}^{\infty} (-1)^m (q;q^2)_m a^m q^{m^2} + 2 \sum_{n=0}^{\infty} \frac{a^n q^{2n^2}}{(aq;q^2)_n}.$$

□

When $a = 1$, the latter series on the right-hand side of Entry 6.5.3 is one of Ramanujan's fifth order mock theta functions, namely $\phi(-q)$.

6.6 The Warnaar Theory

The final objective in this chapter is to prove the fourth identity on page 12 of the lost notebook, Entry 6.6.1. The proof given in [21] is, at best, cumbersome and unenlightening. Warnaar [273] has developed an extended and beautiful theory of partial theta functions. Consequently, he was able to provide a much more coherent proof of Entry 6.6.1.

Warnaar's starting point is the following striking generalization of the Jacobi triple product identity [273, p. 367, equation (1.7)].

Theorem 6.6.1. *For any complex numbers a and b,*

$$(q)_\infty(a)_\infty(b)_\infty\sum_{n=0}^{\infty}\frac{(ab/q)_{2n}q^n}{(q)_n(a)_n(b)_n(ab)_n}$$
$$=1+\sum_{n=1}^{\infty}(-1)^n a^n q^{n(n-1)/2}+\sum_{n=1}^{\infty}(-1)^n b^n q^{n(n-1)/2}. \quad (6.6.1)$$

Thus, Warnaar has transformed the sum of two independent partial theta functions into a single q-hypergeometric series.

We shall not follow Warnaar's original proof of (6.6.1) but shall instead deduce it from the formula for the product of two independent partial theta functions given in the following theorem [38]. But before commencing with this proof, we mention two further related approaches to Theorem 6.6.1 made by A. Berkovich [50]. First, Berkovich deduced (6.6.1) from a formula of G. Gasper and M. Rahman [150], [151, p. 235, equation (8.8.180)] for a product of two ${}_2\phi_1$ series as a sum of two ${}_6\phi_5$ series. Second, he showed that (6.6.1) is equivalent to the identity

$$\sum_{n=1}^{\infty}(-1)^n\frac{x^n-y^n}{x-y}q^{n(n-1)/2}=-(q)_\infty(xq)_\infty(yq)_\infty\sum_{n=0}^{\infty}\frac{(xy)_{2n}q^n}{(q)_n(xq)_n(yq)_n(xy)_n},$$

which had earlier been discovered by A. Schilling and S.O. Warnaar [251, Lemma 4.3 (first line of the proof)].

Theorem 6.6.2. *For any complex numbers a and b,*

$$(q)_\infty(a)_\infty(b)_\infty\sum_{n=0}^{\infty}\frac{(abq^{n-1})_n q^n}{(q)_n(a)_n(b)_n}$$
$$=\left(\sum_{n=0}^{\infty}(-1)^n a^n q^{n(n-1)/2}\right)\left(\sum_{n=0}^{\infty}(-1)^n b^n q^{n(n-1)/2}\right). \quad (6.6.2)$$

As we shall see, identities (6.6.1) and (6.6.2) are equivalent. However, a direct proof of (6.6.2) is a tiny bit smoother than that of (6.6.1), and (6.6.1) is easily deduced from (6.6.2). An elegant combinatorial proof of (6.6.2) has been devised by B. Kim [189].

Proof of Theorem 6.6.2. Let us use the notation

$$[x^n]\sum_{m=0}^{\infty}A_m x^m=A_n.$$

Identity (6.6.2) is equivalent to the assertion that

$$[a^M][b^N](q)_\infty(a)_\infty(b)_\infty \sum_{n=0}^{\infty} \frac{(abq^{n-1})_n q^n}{(q)_n(a)_n(b)_n} = (-1)^{M+N} q^{M(M-1)/2+N(N-1)/2}. \tag{6.6.3}$$

If we denote the left-hand side of (6.6.2) by $L(a,b)$, then noting that $(x)_\infty/(x)_n = (xq^n)_\infty$ and invoking (1.2.2) twice and (1.7.3) once, we see that

$$\begin{aligned}[a^M][b^N]L(a,b) &= (-1)^{M+N}(q)_\infty \\ &\quad\times \sum_{n=0}^{\infty}\sum_{j=0}^{n} \frac{(-1)^j q^{n(1+M+N-j)+j(j-1)/2+(M-j)(M-j-1)/2+(N-j)(N-j-1)/2-j}}{(q)_j(q)_{n-j}(q)_{M-j}(q)_{N-j}}.\end{aligned}$$

Invert the order of summation, replace n by $n+j$, and use the corollary of the q-binomial theorem (1.2.3) to deduce that

$$\begin{aligned}[a^M][b^N]L(a,b) &= (-1)^{M+N}(q)_\infty \\ &\quad\times \sum_{j=0}^{\infty} \frac{(-1)^j q^{j(1+M+N-j)+j(j-1)/2+(M-j)(M-j-1)/2+(N-j)(N-j-1)/2-j}}{(q)_j(q)_{M-j}(q)_{N-j}} \\ &\quad\times \sum_{n=0}^{\infty} \frac{q^{n(1+M+N-j)}}{(q)_n} \\ &= (-1)^{M+N}(q)_\infty \\ &\quad\times \sum_{j=0}^{\infty} \frac{(-1)^j q^{j(1+M+N-j)+j(j-1)/2+(M-j)(M-j-1)/2+(N-j)(N-j-1)/2-j}}{(q)_j(q)_{M-j}(q)_{N-j}(q^{1+M+N-j})_\infty}.\end{aligned}$$

Now use the easily established identity

$$(q)_{n-j} = \frac{(-1)^j (q)_n}{q^{nj-j(j-1)/2}(q^{-n})_j}, \qquad 0 \le j \le n,$$

with $n = M, N, M+N$, and after an enormous amount of simplification, we find that

$$[a^M][b^N]L(a,b) = \frac{(-1)^{M+N}(q)_{M+N} q^{M(M-1)/2+N(N-1)/2}}{(q)_M(q)_N} \tag{6.6.4}$$

$$\times \sum_{j=0}^{\min(M,N)} \frac{(q^{-M})_j(q^{-N})_j q^j}{(q)_j(q^{-M-N})_j}$$

$$= (-1)^{M+N} q^{M(M-1)/2+N(N-1)/2}, \tag{6.6.5}$$

where we applied the second form of the q-Chu–Vandermonde summation theorem (1.3.4). The evaluation (6.6.4) is what we wanted to demonstrate in (6.6.3), and so the proof is complete. □

Proof of (6.6.1). We observe that

$$\begin{aligned}
&(q)_\infty(a)_\infty(b)_\infty \sum_{n=0}^{\infty} \frac{(ab/q)_{2n} q^n}{(q)_n(a)_n(b)_n(ab)_n} \\
&\quad = (q)_\infty(a)_\infty(b)_\infty \sum_{n=0}^{\infty} \frac{(1-ab/q)(abq^n)_{n-1} q^n}{(q)_n(a)_n(b)_n} \\
&\quad = (q)_\infty(a)_\infty(b)_\infty \sum_{n=0}^{\infty} \frac{(1-abq^{n-1}-(ab/q)(1-q^n))(abq^n)_{n-1} q^n}{(q)_n(a)_n(b)_n} \\
&\quad = L(a,b) - ab(q)_\infty(a)_\infty(b)_\infty \sum_{n=1}^{\infty} \frac{(abq^n)_{n-1} q^{n-1}}{(q)_{n-1}(a)_n(b)_n} \\
&\quad = L(a,b) - abL(aq,bq), \qquad (6.6.6)
\end{aligned}$$

where in the penultimate step we replaced n by $n+1$.

On the other hand, by (6.6.2),

$$\begin{aligned}
&L(a,b) - abL(aq,bq) \\
&\quad = \left(\sum_{n=0}^{\infty}(-1)^n a^n q^{n(n-1)/2}\right)\left(\sum_{n=0}^{\infty}(-1)^n b^n q^{n(n-1)/2}\right) \\
&\quad\quad - \left(\sum_{n=1}^{\infty}(-1)^n a^n q^{n(n-1)/2}\right)\left(\sum_{n=1}^{\infty}(-1)^n b^n q^{n(n-1)/2}\right) \\
&\quad = 1 + \sum_{n=1}^{\infty}(-1)^n a^n q^{n(n-1)/2} + \sum_{n=1}^{\infty}(-1)^n b^n q^{n(n-1)/2}. \qquad (6.6.7)
\end{aligned}$$

If we employ (6.6.7) in (6.6.6), we complete the proof of (6.6.1). □

We now restrict our considerations of Warnaar's work to those portions necessary to prove Entry 6.6.1 below. We point out, in Warnaar's words, that this is "just the tip of the iceberg." Many further results are obtained by Warnaar in his 33-page paper [273], including applications of Bailey pairs to produce elegant multiple series identities for partial theta functions. Warnaar's theorem below can be found in [273, p. 370, equation (3.7)].

Theorem 6.6.3. *If $abc \neq 0$,*

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(q^{n+1})_n (q^2/(bc))_n q^n}{(a)_{n+1}(q/a)_n(q^2/b)_n(q^2/c)_n} - \sum_{n=0}^{\infty} \frac{(b)_n(c)_n}{(q^2/b)_n(q^2/c)_n}\left(\frac{aq^2}{bc}\right)^n \\
&\quad = \frac{1}{(q)_\infty(a)_\infty(q/a)_\infty} \sum_{n=1}^{\infty}(-1)^n a^n q^{n(n+1)/2} \\
&\quad\quad \times \sum_{r=-\infty}^{\infty} \frac{(-1)^r (b)_r(c)_r q^{r(r-1)/2}}{(q^2/b)_r(q^2/c)_r}\left(\frac{q^{n+3}}{bc}\right)^r.
\end{aligned}$$

Before commencing our proof, we note that if we replace a by $-a$, set $b=-q$, and let $c\to\infty$ in Theorem pt.t4, we deduce Entry 6.3.11.

Proof. We follow the development in [273, pp. 369–370]. First, we replace a by aq^{r+1} and b by bq^r in (6.6.1). After multiplying the numerator and denominator on the right-hand side by $(a)_{r+1}(b)_r(ab)_{2r+1}$, we find that

$$\begin{aligned}1+&\sum_{n=1}^{\infty}(-1)^n q^{n(n-1)/2}\left((aq^{r+1})^n+(bq^r)^n\right)\\ &=q^{-r}(q)_\infty(a)_\infty(b)_\infty\frac{1-abq^{2r}}{1-ab}\sum_{n=r}^{\infty}\frac{(ab)_{2n}q^n}{(q)_{n-r}(abq)_{n+r}(a)_{n+1}(b)_n}.\end{aligned}\tag{6.6.8}$$

Second, observe from the Jacobi triple product identity (1.4.8) that

$$(q/b)^r(b)_\infty(q/b)_\infty(q)_\infty=\sum_{n=-\infty}^{\infty}b^nq^{(n+r)(n+r-1)/2+r}.\tag{6.6.9}$$

If we now add and subtract the series

$$\sum_{n=1}^{\infty}(-1)^{n+r}a^nq^{(n+r)(n+r+1)/2}$$

to (6.6.9) and rearrange, we find that

$$\begin{aligned}(-1)^rq^{r(r+1)/2}&\left(1+\sum_{n=1}^{\infty}(-1)^nq^{n(n-1)/2}\left((aq^{r+1})^n+(bq^r)^n\right)\right)\\ &=(q/b)^r(b)_\infty(q/b)_\infty(q)_\infty\\ &\quad+\sum_{n=1}^{\infty}(-1)^{n+r}\left(a^nq^{(n+r)(n+r+1)/2}-(q/b)^nq^{(n-r)(n-r-1)/2}\right).\end{aligned}$$

Hence, using this last identity in (6.6.8), we deduce that

$$\begin{aligned}&(q/b)^r(b)_\infty(q/b)_\infty(q)_\infty\\ &\quad+\sum_{n=1}^{\infty}(-1)^{n+r}\left(a^nq^{(n+r)(n+r+1)/2}-(q/b)^nq^{(n-r)(n-r-1)/2}\right)\\ &=(-1)^rq^{r(r-1)/2}(q)_\infty(a)_\infty(b)_\infty\frac{1-abq^{2r}}{1-ab}\sum_{n=r}^{\infty}\frac{(ab)_{2n}q^n}{(q)_{n-r}(abq)_{n+r}(a)_{n+1}(b)_n}.\end{aligned}$$

We now multiply both sides of this last identity by a sequence f_r, to be specified later, and sum over all nonnegative integers r. Hence, assuming that f_r satisfies suitable conditions for convergence, inverting the order of summation, and dividing both sides of the resulting identity by $(q)_\infty(a)_\infty(b)_\infty$, we find that

$$\sum_{n=0}^{\infty} \frac{(ab)_{2n} q^n}{(a)_{n+1}(b)_n} \sum_{r=0}^{n} \frac{(-1)^r f_r q^{r(r-1)/2}(1-abq^{2r})}{(q)_{n-r}(ab)_{n+r+1}} - \frac{(q/b)_\infty}{(a)_\infty} \sum_{r=0}^{\infty} (q/b)^r f_r$$
$$= \frac{(-1)^{n+r} q^{(n+r)(n+r+1)/2}}{(q)_\infty (a)_\infty (b)_\infty} \sum_{n=1}^{\infty} \left(a^n \sum_{r=0}^{\infty} f_r + (q/b)^n \sum_{r=-\infty}^{-1} f_{-r-1} \right). \tag{6.6.10}$$

We now specialize (6.6.10) by replacing b by q/a and then taking

$$f_r := \frac{(b)_r (c)_r}{(q^2/b)_r (q^2/c)_r} \left(\frac{q^2}{bc} \right)^r. \tag{6.6.11}$$

Observe that for each nonnegative integer r,

$$(b)_{-r-1} = \frac{(-1)^{r+1} q^{(r+1)(r+2)/2}}{b^{r+1}(q/b)_{r+1}}.$$

Replacing b by q^2/b above, we find that

$$(q^2/b)_{-r-1} = \frac{(-1)^{r+1} q^{(r+1)(r+2)/2}}{(q^2/b)^{r+1}(b/q)_{r+1}}.$$

Dividing the penultimate equality by the last equality and simplifying, we find that

$$\frac{(b)_{-r-1}}{(q^2/b)_{-r-1}} \left(\frac{q}{b} \right)^{-r-1} = -\frac{(b)_r}{(q^2/b)_r} \left(\frac{q}{b} \right)^r.$$

Replacing b by c above and multiplying the two equalities together, we deduce that

$$f_{-r-1} = f_r. \tag{6.6.12}$$

Furthermore, with f_r defined by (6.6.11), we find that

$$S := \sum_{r=0}^{n} \frac{(-1)^r f_r q^{r(r-1)/2}(1-q^{2r+1})}{(q)_{n-r}(q)_{n+r+1}}$$
$$= \frac{1}{(q)_n (q)_{n+1}} \sum_{r=0}^{n} \frac{(b)_r (c)_r (q^{-n})_r (q^2/(bc))^r q^{nr}(1-q^{2r+1})}{(q^2/b)_r (q^2/c)_r (q^{n+2})_r}. \tag{6.6.13}$$

We now apply (4.1.3) to the right-hand side of (6.6.13) with $\beta = \alpha q/\gamma$, and then with $\alpha = q$, $N = n$, $\delta = b$, and $\epsilon = c$. Accordingly, upon simplification, we conclude that

$$S = \frac{(q^2/(bc))_n}{(q)_n (q^2/b)_n (q^2/c)_n}. \tag{6.6.14}$$

Lastly, we utilize (6.6.12) and put (6.6.11) and (6.6.14) into (6.6.10). After simplification, we deduce the identity of Theorem 6.6.3. □

Entry 6.6.1 (p. 12). *For any complex number $a \neq 0$,*

$$\sum_{n=0}^{\infty} \frac{(q^{n+1})_n q^n}{(-aq)_n(-q/a)_n} = (1+a)\sum_{n=0}^{\infty}(-1)^n a^n q^{n(n+1)} - \frac{a}{(-aq)_\infty(-q/a)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{3n} q^{3n^2+2n}(1+aq^{2n+1}).$$

Proof. In Theorem 6.6.3, replace a by $-a$, multiply both sides by $(1+a)$, and let b and c tend to ∞. Consequently,

$$\begin{aligned}
T &:= \sum_{n=0}^{\infty} \frac{(q^{n+1})_n q^n}{(-aq)_n(-q/a)_n} - (1+a)\sum_{r=0}^{\infty}(-1)^r a^r q^{r(r+1)} \\
&= \frac{1}{(q)_\infty(-aq)_\infty(-q/a)_\infty}\sum_{n=1}^{\infty} a^n q^{n(n+1)/2} \sum_{r=-\infty}^{\infty}(-1)^r q^{3r(r-1)/2+(n+3)r} \\
&= \frac{1}{(q)_\infty(-aq)_\infty(-q/a)_\infty}\sum_{\nu=-3}^{-1}\sum_{n=1}^{\infty} a^{3n+\nu} q^{(3n+\nu)(3n+\nu+1)/2} \\
&\quad \times \sum_{r=-\infty}^{\infty}(-1)^r q^{3r(r-1)/2+(3n+\nu+3)r} \\
&= \frac{1}{(q)_\infty(-aq)_\infty(-q/a)_\infty}\sum_{\nu=-3}^{-1}\sum_{n=1}^{\infty} a^{3n+\nu} q^{\nu(\nu+1)/2+3n^2+2n\nu} \\
&\quad \times \sum_{r=-\infty}^{\infty}(-1)^{r-n} q^{3r(r+1)/2+r\nu}.
\end{aligned}$$

Recall from (1.4.12) that $f(-1,q^3)=0$. Thus, the sum on r vanishes when $\nu=0$, and otherwise the sum is equal to $(-1)^n(q)_\infty$ by (1.4.8). Thus, we find that

$$\begin{aligned}
T &= \frac{1}{(-aq)_\infty(-q/a)_\infty}\sum_{\nu=-2}^{-1}\sum_{n=1}^{\infty}(-1)^n a^{3n+\nu} q^{\nu(\nu+1)/2+3n^2+2n\nu} \\
&= -\frac{1}{(-aq)_\infty(-q/a)_\infty}\left(\sum_{n=0}^{\infty}(-1)^n a^{3n+1} q^{3n^2+2n} - \sum_{n=0}^{\infty}(-1)^n a^{3n+2} q^{3n^2+4n+1}\right) \\
&= -\frac{a}{(-aq)_\infty(-q/a)_\infty}\sum_{n=0}^{\infty}(-1)^n a^{3n} q^{3n^2+2n}(1+aq^{2n+1}).
\end{aligned}$$

This therefore completes the proof. ∎

7

Special Identities

7.1 Introduction

A few of Ramanujan's identities do not fit naturally into any of the previous chapters. In this chapter, we have gathered two groups of such results. The first four identities to be examined have previously been proved [20] by relating them to the theory of Durfee rectangles [13]. We provide an alternative development based on functional equations in Section 7.2.

The three identities in Section 7.3 are among the more surprising identities in the lost notebook. They were first proved in [25]; however, the proofs there provide no significant insight into the reasons for their existence. In [36], the fundamental idea lying behind these results was exposed in Proposition 2.1 (Entry 7.2.4 below). Subsequently, Andrews and P. Freitas [35], G.H. Coogan and K. Ono [138], and J. Lovejoy and Ono [213] have exploited this method and given some further interesting applications to evaluations of L-series. In [35], Proposition 2.1 of [36] was placed in the context of Abel's Lemma and greatly generalized. S.H. Chan [119] derived an identity in the same spirit of Ramanujan's identity, but with arbitrarily many q-products. To effect this, he employed Sears's general transformation formula to derive a power series identity with the requisite q-products, and then proceeded on the same path as previous authors by applying the operator $\lim_{x\to 1^-} \frac{d}{dx}$ to both sides of his identity.

Section 7.4 concludes the chapter with a couple of innocent formulas that require two completely different representations of the very well-poised ${}_{10}\phi_9$ to effect their proofs. This seeming mismatch between simplicity of statement and complexity of proof is among the more mysterious aspects of the lost notebook; hence this section is named *Innocents Abroad*.

G.E. Andrews, B.C. Berndt, *Ramanujan's Lost Notebook: Part II*,
DOI 10.1007/978-0-387-77766-5_8,

7.2 Generalized Modular Relations

In his enigmatic list [85], [32] of 40 modular relations for the Rogers–Ramanujan functions

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} \qquad \text{and} \qquad H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n},$$

Ramanujan asserted that

$$G(q)G(q^4) + qH(q)H(q^4) = \frac{\varphi(q)}{(q^2;q^2)_\infty}. \tag{7.2.1}$$

G.N. Watson, in addition to proving (7.2.1) in [277], also proved in [280] that

$$G(-q)\varphi(q) - G(q)\varphi(-q) = 2qH(q^4)\psi(q^2) \tag{7.2.2}$$

and

$$H(-q)\varphi(q) + H(q)\varphi(-q) = 2G(q^4)\psi(q^2). \tag{7.2.3}$$

These three modular relations all turn out to be specializations of formulas from the lost notebook, as we shall see.

Entry 7.2.1 (p. 27). *For any complex numbers a and b,*

$$\sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2}}{(-aq)_n(-bq)_n} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} b^n q^{2n^2}}{(-aq)_n(-bq)_{2n}} + \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1} b^{n+1} q^{2n^2+3n+1}}{(-aq)_n(-bq)_{2n+1}}. \tag{7.2.4}$$

Entry 7.2.1 was proved in [13], where the proof was based on Durfee rectangles. We first give this proof and then provide a different proof.

First Proof of Entry 7.2.1. Replacing a and b by $-a$ and $-b$, respectively, we prove the equivalent identity

$$\sum_{n=0}^{\infty} \frac{a^n b^n q^{n^2}}{(aq)_n(bq)_n} = \sum_{n=0}^{\infty} \frac{a^{2n} b^n q^{2n^2}}{(aq)_n(bq)_{2n}} + \sum_{n=0}^{\infty} \frac{a^{2n+1} b^{n+1} q^{2n^2+3n+1}}{(aq)_n(bq)_{2n+1}}. \tag{7.2.5}$$

When $a = 1$, the coefficient of $b^m q^n$ on the left-hand side of (7.2.5) is the number of partitions of n with exactly m parts. For general a, we further see that the coefficient of $b^m a^r q^n$ is the number of partitions of n into m parts with largest part equal to r.

Our objective next is to identify the right-hand side of (7.2.5) as precisely the same generating function. Consider a Ferrers rectangle of n rows and $2n$ columns of nodes. To these $2n^2$ nodes we attach a partition π_1 (to be read by columns) on the right side with at most n dots in each column, while below we attach a partition π_2 with at most $2n$ dots in each row. We note that the total number of parts in the entire partition is

$$n + \text{ the number of parts of } \pi_2,$$

while the largest part of the partition is

$$2n + \text{ the number of columns of } \pi_1.$$

The generating function for all such partitions with this Durfee rectangle is

$$\frac{b^n a^{2n} q^{2n^2}}{(aq)_n (bq)_{2n}}, \tag{7.2.6}$$

where the exponent of b gives the number of parts of the partition, and the exponent of a yields the largest part of the partition. If we sum over all n, $0 \leq n < \infty$, it appears that we obtain all such partitions and thus obtain a new representation for the left side of (7.2.5). This is not the case, however, because we have omitted from consideration a large class of partitions, namely, all partitions π with a node in the $(n+1)$st row and $(2n+1)$st column, and which are not counted in partitions associated with the next-largest Durfee rectangle, i.e., the one of size $(n+1) \times (2n+2)$. These omitted partitions attached to the $n \times 2n$ Durfee rectangle are generated by

$$\frac{b^{n+1} a^{2n+1} q^{(n+1)(2n+1)}}{(aq)_n (bq)_{2n+1}}. \tag{7.2.7}$$

If we now sum both (7.2.6) and (7.2.7) over n, $0 \leq n < \infty$, then we obtain the generating function for all partitions in which the coefficient of $b^m a^r q^n$ is the number of partitions of n into m parts with the largest part equal to r. Consequently, the two sides of (7.2.5) are equal. □

Second Proof of Entry 7.2.1. Let us denote the left-hand side of this entry by $f_L(a,b)$ and the right-hand side by $f_R(a,b)$, where

$$f_R(a,b) = T_1(a,b) + T_2(a,b), \tag{7.2.8}$$

with T_1 and T_2 being, respectively, the two sums that appear on the right side of (7.2.4). We note that $f_L(a,b)$ and $f_R(a,b)$ are analytic in a and b around $(0,0)$, where each takes the value 1. Furthermore,

$$\begin{aligned} f_L(a,b) &= 1 + \sum_{n=0}^{\infty} \frac{a^{n+1} b^{n+1} q^{n^2+2n+1}}{(-aq)_{n+1}(-bq)_{n+1}} \\ &= 1 + \frac{abq}{(1+aq)(1+bq)} f_L(aq, bq), \end{aligned} \tag{7.2.9}$$

and by iteration we see that $f_L(a,b)$ is uniquely determined by (7.2.9) and analyticity at $(0,0)$. So to conclude the proof of this identity, we only need to show that $f_R(a,b)$ satisfies the same functional equation as (7.2.9).

Now,

$$
\begin{aligned}
T_1(a,b) &= 1 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1} a^{2n+2} b^{n+1} q^{2n^2+4n+2}}{(-aq)_{n+1}(-bq)_{2n+2}} \\
&= 1 - \frac{abq}{(1+aq)(1+bq)} \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1} b^n q^{2n^2+4n+1}}{(-aq^2)_n(-bq^2)_{2n+1}}
\end{aligned}
$$

and

$$
\begin{aligned}
T_2(a,b) &= \frac{abq}{(1+aq)(1+bq)} \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} b^n q^{2n^2+3n}(1+aq^{n+1})}{(-aq^2)_n(-bq^2)_{2n}} \\
&= \frac{abq}{(1+aq)(1+bq)} T_1(aq,bq) \\
&\quad + \frac{a^2 b}{(1+aq)(1+bq)} \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} b^n q^{2n^2+4n+2}}{(-aq^2)_n(-bq^2)_{2n}}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
f_R(a,b) &= 1 - \frac{abq}{(1+aq)(1+bq)} \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1} b^n q^{2n^2+4n+1}}{(-aq^2)_n(-bq^2)_{2n+1}} \\
&\quad + \frac{abq}{(1+aq)(1+bq)} T_1(aq,bq) \\
&\quad + \frac{a^2 b}{(1+aq)(1+bq)} \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} b^n q^{2n^2+4n+2}}{(-aq^2)_n(-bq^2)_{2n}} \\
&= 1 + \frac{abq}{(1+aq)(1+bq)} T_1(aq,bq) + \frac{abq}{(1+aq)(1+bq)} T_2(aq,bq) \\
&= 1 + \frac{abq}{(1+aq)(1+bq)} f_R(aq,bq).
\end{aligned}
$$

□

There is a further representation of f_L and f_R required subsequently, namely,

$$
f_L(a,b) = f_R(a,b) = f_0(a,b) := 1 - b \sum_{n=1}^{\infty} \frac{(-1)^n a^n q^n}{(-bq)_n}. \tag{7.2.10}
$$

Clearly $f_0(0,0) = 1$ and $f_0(a,b)$ is continuous in a neighborhood of $(0,0)$. Furthermore,

$$
\begin{aligned}
f_0(a,b) &= 1 + \frac{abq}{1+bq} + \frac{b}{1+bq} \sum_{n=1}^{\infty} \frac{(-1)^n a^{n+1} q^{n+1}}{(-bq^2)_n} \\
&= 1 + \frac{abq}{1+bq} + \frac{a}{1+bq} (1 - f_0(a,bq)) \\
&= 1 + a - \frac{a}{1+bq} f_0(a,bq).
\end{aligned} \tag{7.2.11}
$$

On the other hand,

$$\begin{aligned} f_0(a,b)+bf_0(aq,b) &= 1+b-b\sum_{n=1}^{\infty}\frac{(-1)^n a^n q^n(1+bq^n)}{(-bq)_n} \\ &= 1+b-b\sum_{n=1}^{\infty}\frac{(-1)^n a^n q^n}{(-bq)_{n-1}} \\ &= 1+b+b\sum_{n=0}^{\infty}\frac{(-1)^n a^{n+1} q^{n+1}}{(-bq)_n} \\ &= 1+b+baq+aq(1-f_0(a,b)). \end{aligned} \tag{7.2.12}$$

Hence,

$$f_0(a,b)=1+b-\frac{b}{1+aq}f_0(aq,b). \tag{7.2.13}$$

We now replace b by bq in (7.2.13) and then substitute the resulting identity into (7.2.11) to deduce that

$$\begin{aligned} f_0(a,b) &= 1+a-\frac{a}{1+bq}\left(1+bq-\frac{bq}{1+aq}f(aq,bq)\right) \\ &= 1+\frac{abq}{(1+aq)(1+bq)}f_0(aq,bq). \end{aligned}$$

Thus, $f_0(a,b)$ satisfies the same functional equation and initial conditions as $f_L(a,b)$ and $f_R(a,b)$. Thus (7.2.10) is proved.

Finally, we also require the values

$$f_L(a,-1)=f_R(a,-1)=f_0(a,-1)=\frac{1}{(-aq)_\infty}. \tag{7.2.14}$$

These immediately follow from the fact that

$$f_0(a,-1)=1+\sum_{n=1}^{\infty}\frac{(-1)^n a^n q^n}{(q)_n}=\sum_{n=0}^{\infty}\frac{(-aq)^n}{(q)_n}=\frac{1}{(-aq)_\infty}, \tag{7.2.15}$$

by (1.2.3).

Entry 7.2.2 (p. 27). *If a and b are any complex numbers, except that $a \neq 0$, then*

$$\begin{aligned} &\sum_{n=0}^{\infty}\frac{a^n b^n q^{n^2/4}}{(q)_n}\sum_{m=0}^{\infty}\frac{a^{-2m}q^{m^2}}{(bq)_m}+\sum_{n=0}^{\infty}\frac{a^n b^n q^{(n+1)^2/4}}{(q)_n}\sum_{m=0}^{\infty}\frac{a^{-2m-1}q^{m^2+m}}{(bq)_m} \\ &\qquad = \frac{1}{(bq)_\infty}\sum_{n=-\infty}^{\infty}a^n q^{n^2/4}-(1-b)\sum_{n=1}^{\infty}a^n q^{n^2/4}\sum_{j=0}^{n-1}\frac{b^j}{(q)_j}. \end{aligned}$$

Proof. Noting that $1/(q)_{-n} = 0$ for $n > 0$, we may equate coefficients of a^N, $-\infty < N < \infty$, on each side of the desired identity. Hence, we must prove that, for every integer N,

$$\sum_{m=0}^{\infty} \frac{b^{N+2m} q^{(N+2m)^2/4+m^2}}{(q)_{N+2m}(bq)_m} + \sum_{m=0}^{\infty} \frac{b^{N+2m+1} q^{(N+2m+2)^2/4+m^2+m}}{(q)_{N+2m+1}(bq)_m}$$
$$= \begin{cases} \dfrac{q^{N^2/4}}{(bq)_\infty}, & \text{if } N \le 0, \\ \dfrac{q^{N^2/4}}{(bq)_\infty} - (1-b)q^{N^2/4} \displaystyle\sum_{j=0}^{N-1} \frac{b^j}{(q)_j}, & \text{if } N > 0. \end{cases} \tag{7.2.16}$$

We distinguish three cases in the proof of (7.2.16).

Case 1. $N = -2\nu$, where ν is a nonnegative integer. Replace m by $m + \nu$ throughout (7.2.16). Accordingly, the left-hand side of (7.2.16) becomes, with the use of (7.2.15),

$$\sum_{m=0}^{\infty} \frac{b^{2m} q^{2m^2+2m\nu+\nu^2}}{(q)_{2m}(bq)_{m+\nu}} + \sum_{m=0}^{\infty} \frac{b^{2m+1} q^{2m^2+3m+1+2m\nu+\nu^2+\nu}}{(q)_{2m+1}(bq)_{m+\nu}}$$
$$= \frac{q^{\nu^2}}{(bq)_\nu} f_R(-bq^\nu, -1)$$
$$= \frac{q^{\nu^2}}{(bq)_\nu} \frac{1}{(bq^{\nu+1})_\infty} = \frac{q^{\nu^2}}{(bq)_\infty} = \frac{q^{N^2/4}}{(bq)_\infty},$$

and thus (7.2.16) is established in this case.

Case 2. $N = -2\nu-1$, where ν is a nonnegative integer. Replace m by $m+\nu+1$ in the first sum in (7.2.16) and by $m + \nu$ in the second. So, the left-hand side of (7.2.16) becomes

$$\sum_{m=0}^{\infty} \frac{b^{2m+1} q^{(2m+1)^2/4+(m+\nu+1)^2}}{(q)_{2m+1}(bq)_{m+\nu+1}} + \sum_{m=0}^{\infty} \frac{b^{2m} q^{(2m+1)^2/4+(m+\nu)^2+m+\nu}}{(q)_{2m}(bq)_{m+\nu}}$$
$$= \frac{q^{\nu^2+\nu+1/4}}{(bq)_\nu} \left\{ \sum_{m=0}^{\infty} \frac{(bq^\nu)^{2m} q^{2m^2}(1-(1-q^{2m}))}{(q)_{2m}(bq^{\nu+1})_m} \right.$$
$$\left. + \sum_{m=0}^{\infty} \frac{(bq^\nu)^{2m+1} q^{2m^2+3m+1}}{(q)_{2m+1}(bq^{\nu+1})_m} \left(1 + \frac{bq^{m+\nu+1}}{1-bq^{m+\nu+1}}\right) \right\}$$
$$= \frac{q^{(\nu+1/2)^2}}{(bq)_\nu} f_R(-bq^\nu, -1) + \frac{q^{(\nu+1/2)^2}}{(bq)_\nu} \left\{ -\sum_{m=1}^{\infty} \frac{(bq^\nu)^{2m} q^{2m^2}}{(q)_{2m-1}(bq^{\nu+1})_m} \right.$$
$$\left. + \sum_{m=0}^{\infty} \frac{(bq^\nu)^{2m+2} q^{2(m+1)^2}}{(q)_{2m+1}(bq^{\nu+1})_{m+1}} \right\}$$
$$= \frac{q^{(\nu+1/2)^2}}{(bq)_\nu} \frac{1}{(bq^{\nu+1})_\infty} = \frac{q^{N^2/4}}{(bq)_\infty},$$

by (7.2.10), (7.2.14), and the fact that in the penultimate expression, the series inside curly brackets cancel when m is replaced by $m+1$ in the first sum. Hence, (7.2.16) is established in the second case.

Case 3. $N > 0$. Using (1.2.3), we find that the right-hand side of (7.2.16) is equal to

$$\begin{aligned}
\frac{q^{N^2/4}}{(bq)_\infty} - (1-b)q^{N^2/4}\sum_{j=0}^{N-1}\frac{b^j}{(q)_j} &= q^{N^2/4}\left(\frac{1}{(bq)_\infty} - \sum_{j=0}^{N-1}\frac{b^j}{(q)_j} + \sum_{j=1}^{N}\frac{b^j}{(q)_{j-1}}\right)\\
&= q^{N^2/4}\left(\frac{1}{(bq)_\infty} - \sum_{j=0}^{N}\frac{b^jq^j}{(q)_j} + \frac{b^N}{(q)_N}\right)\\
&= q^{N^2/4}\left(\sum_{j=N+1}^{\infty}\frac{b^jq^j}{(q)_j} + \frac{b^N}{(q)_N}\right)\\
&= \frac{q^{N^2/4}b^N}{(q)_N}\left(1 + \sum_{j=N+1}^{\infty}\frac{b^{j-N}q^j}{(q^{N+1})_{j-N}}\right)\\
&= \frac{q^{N^2/4}b^N}{(q)_N}\left(1 + \sum_{j=1}^{\infty}\frac{b^jq^{N+j}}{(q^{N+1})_j}\right)\\
&= \frac{q^{N^2/4}b^N}{(q)_N}f_0(-b,-q^N), \qquad (7.2.17)
\end{aligned}$$

by (7.2.10). On the other hand, the left-hand side of (7.2.16) is

$$\begin{aligned}
&\frac{q^{N^2/4}b^N}{(q)_N}\left\{\sum_{m=0}^{\infty}\frac{b^{2m}q^{Nm}q^{2m^2}}{(q^{N+1})_{2m}(bq)_m} + \sum_{m=0}^{\infty}\frac{b^{2m+1}q^{N(m+1)}q^{2m^2+3m+1}}{(q^{N+1})_{2m+1}(bq)_m}\right\}\\
&\qquad = \frac{q^{N^2/4}b^N}{(q)_N}f_R(-b,-q^N), \qquad (7.2.18)
\end{aligned}$$

by (7.2.4) and (7.2.8). In light of the fact that $f_R(a,b) = f_0(a,b)$, we see, from (7.2.17) and (7.2.18), that the two sides of (7.2.16) are identical in this final case. □

Entry 7.2.3 (p. 26). *For $a \neq 0$,*

$$\begin{aligned}
&\left(\sum_{n=0}^{\infty}\frac{a^nq^{n^2/4}}{(q)_n}\right)\left(\sum_{n=0}^{\infty}\frac{a^{-2n}q^{n^2}}{(q)_n}\right) + \left(\sum_{n=0}^{\infty}\frac{a^nq^{(n+1)^2/4}}{(q)_n}\right)\left(\sum_{n=0}^{\infty}\frac{a^{-2n-1}q^{n^2+n}}{(q)_n}\right)\\
&\qquad = \frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}a^nq^{n^2/4}.
\end{aligned}$$

Proof. Set $b = 1$ in Entry 7.2.2. □

In the middle of page 26 of the lost notebook, we find the claim, upon changing to more contemporary notation,

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2/(2s)}}{(q)_n} \sum_{n=0}^{\infty} \frac{a^{-ns} q^{n^2 s/2}}{(q)_n} \qquad \text{as } q \to 1??$$

We note that this is the first expression on the left-hand side of Entry 7.2.3 when $s = 2$. Ramanujan provides no indication either why this is of interest for arbitrary s or what the asymptotics should be.

Entry 7.2.4 (p. 26). *For $a \neq 0$,*

$$\left(\sum_{n=-\infty}^{\infty} a^n q^{n^2/4}\right)\left(\sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n^2/4}}{(q)_n}\right) \\ - \left(\sum_{n=-\infty}^{\infty} (-1)^n a^n q^{n^2/4}\right)\left(\sum_{n=0}^{\infty} \frac{a^n q^{n^2/4}}{(q)_n}\right) = 2(q)_\infty \sum_{n=0}^{\infty} \frac{a^{-2n-1} q^{(2n+1)^2/4}}{(q)_n}.$$

Proof. Clearly, the left-hand side of this identity is an odd function of a. Consequently, the coefficients of all even powers of a are equal to 0.

The coefficient of a^{-2N-1} on the left-hand side is

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(-1)^n q^{n^2/4+(-2N-1-n)^2/4}}{(q)_n} - \sum_{n=0}^{\infty} \frac{(-1)^{-2N-1-n} q^{n^2/4+(-2N-1-n)^2/4}}{(q)_n} \\ &\qquad = 2\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2+Nn+(2N+1)^2/4}}{(q)_n} \\ &\qquad = 2q^{(2N+1)^2/4}(q^{N+1})_\infty, \end{aligned}$$

by (1.2.4). We see that we have obtained the coefficient of a^{-2N-1} on the right-hand side of Entry 7.2.4, and so the proof is complete. □

Entry 7.2.5 (p. 30). *If $a \neq 0$, then*

$$\left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty} \frac{(-1)^n (b/a)^n q^{n(n+1)/2}}{(-bq)_n} = \frac{1}{a}\sum_{n=0}^{\infty} \frac{(-1)^n (b/a)^n q^{n(n+1)/2}}{(-aq)_n(-bq)_{2n+1}} \\ + \sum_{n=0}^{\infty} \frac{(-1)^n (b/a)^n q^{n(n+3)/2}}{(-aq)_n(-bq)_{2n}}. \tag{7.2.19}$$

Proof. Multiply both sides of (7.2.19) by $(-bq)_\infty$ to obtain the equivalent identity

$$\left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty} (-1)^n (b/a)^n q^{n(n+1)/2}(-bq^{n+1})_\infty$$

$$
\begin{aligned}
&= \frac{1}{a}\sum_{n=0}^{\infty}\frac{(-1)^n(b/a)^n q^{n(n+1)/2}}{(-aq)_n}(-bq^{2n+2})_\infty \\
&\quad + \sum_{n=0}^{\infty}\frac{(-1)^n(b/a)^n q^{n(n+3)/2}}{(-aq)_n}(-bq^{2n+1})_\infty.
\end{aligned}
$$

If we now expand each infinite product appearing above by (1.2.4), and then compare coefficients of b^N on each side, we find that the proof of Entry 7.2.5 is reduced to proving that, for each nonnegative integer N,

$$
\left(1+\frac{1}{a}\right)\sum_{n=0}^{N}\frac{(-1)^n a^{-n}}{(q)_{N-n}} = \sum_{n=0}^{N}\frac{(-1)^n a^{-n} q^{n(N-n)}(q^n + a^{-1}q^{N-n})}{(-aq)_n (q)_{N-n}},
$$

or equivalently,

$$
L_N := \sum_{n=0}^{N}\frac{(-1)^n a^{-n}}{(q)_{N-n}} = \sum_{n=0}^{N}\frac{(-1)^n a^{-n+1} q^{n(N-n)}(q^n + a^{-1}q^{N-n})}{(-a)_{n+1}(q)_{N-n}} =: R_N. \tag{7.2.20}
$$

Clearly,

$$
L_0 = R_0 = 1. \tag{7.2.21}
$$

Now,

$$
\begin{aligned}
L_N &= \sum_{n=0}^{N}\frac{(-1)^n a^{-n}}{(q)_{N-n}} = \sum_{n=0}^{N}\frac{(-1)^{N-n} a^{-N+n}}{(q)_n} \\
&= -\frac{1}{a}\sum_{n=0}^{N-1}\frac{(-1)^{N-1-n} a^{-N+1+n}}{(q)_n} + \frac{1}{(q)_N} \\
&= -\frac{1}{a}L_{N-1} + \frac{1}{(q)_N}.
\end{aligned} \tag{7.2.22}
$$

On the other hand,

$$
\begin{aligned}
R_N + a^{-1}R_{N-1} &= \sum_{n=0}^{N}\frac{(-1)^n a^{-n+1} q^{n(N-n)+n}}{(-a)_{n+1}(q)_{N-n}} + \sum_{n=0}^{N}\frac{(-1)^n a^{-n} q^{(n+1)(N-n)}}{(-a)_{n+1}(q)_{N-n}} \\
&\quad + \sum_{n=0}^{N-1}\frac{(-1)^n a^{-n} q^{n(N-n)}}{(-a)_{n+1}(q)_{N-n-1}} + \sum_{n=0}^{N-1}\frac{(-1)^n a^{-n-1} q^{(n+1)(N-n-1)}}{(-a)_{n+1}(q)_{N-n-1}} \\
&= \sum_{n=0}^{N}\frac{(-1)^n a^{-n+1} q^{n(N-n)+n}}{(-a)_{n+1}(q)_{N-n}} + \sum_{n=0}^{N}\frac{(-1)^n a^{-n} q^{n(N-n)}}{(-a)_{n+1}(q)_{N-n}} \\
&\quad + \sum_{n=0}^{N-1}\frac{(-1)^n a^{-n-1} q^{(n+1)(N-n-1)}}{(-a)_{n+1}(q)_{N-n-1}},
\end{aligned}
$$

where we used the fact that $1/(q)_{-1} = 0$ and where we added the second and third sums term by term. Adding the first and second sums term by term, we find that

$$\begin{aligned}R_N + a^{-1}R_{N-1} &= \sum_{n=0}^{N} \frac{(-1)^n a^{-n} q^{n(N-n)}}{(-a)_n (q)_{N-n}} + \sum_{n=0}^{N-1} \frac{(-1)^n a^{-n-1} q^{(n+1)(N-n-1)}}{(-a)_{n+1}(q)_{N-n-1}} \\ &= \frac{1}{(q)_N} + \sum_{n=0}^{N-1} \frac{(-1)^{n+1} a^{-n-1} q^{(n+1)(N-n-1)}}{(-a)_{n+1}(q)_{N-n-1}} \\ &\quad + \sum_{n=0}^{N-1} \frac{(-1)^n a^{-n-1} q^{(n+1)(N-n-1)}}{(-a)_{n+1}(q)_{N-n-1}} \\ &= \frac{1}{(q)_N}. \end{aligned} \tag{7.2.23}$$

Thus, since $L_0 = R_0$ by (7.2.21), and since both sides of (7.2.20) satisfy the same inhomogeneous first-order recurrence by (7.2.22) and (7.2.23), we can conclude that

$$L_N = R_N.$$

This therefore proves (7.2.20), and thus the proof of Entry 7.2.5 is complete. □

In concluding this section, we note that (7.2.1) follows immediately from Entry 7.2.2 by replacing q by q^4, setting $a = b = 1$, and invoking the two formulas

$$G(q) = (-q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n}$$

and

$$H(q) = (-q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4;q^4)_n},$$

both due to L.J. Rogers [248]. Identity (7.2.2) follows from Entry 7.2.3 when we replace a by 1 and q by q^4, and then multiply both sides by $(-q^2;q^2)_\infty$. Identity (7.2.3) follows from Entry 7.2.3 if we replace q by q^4, then set $a = q^2$, and lastly multiply both sides by $q(-q^2;q^2)_\infty$.

7.3 Extending Abel's Lemma

The main results in this section, Entries 7.3.2 and 7.3.3, are rather natural corollaries of the following generalization of Abel's Lemma [36, p. 403].

Theorem 7.3.1. *Suppose that*

$$f(z) := \sum_{n=0}^{\infty} \alpha_n z^n$$

is analytic for $|z| < 1$*. Let* α *be a complex number for which*

$$\sum_{n=0}^{\infty} |\alpha - \alpha_n| < +\infty, \tag{7.3.1}$$

$$\lim_{n\to+\infty} n(\alpha - \alpha_n) = 0. \tag{7.3.2}$$

Then

$$\lim_{z\to 1^-} \frac{d}{dz}(1-z)f(z) = \sum_{n=0}^{\infty}(\alpha - \alpha_n).$$

Proof. We recall Abel's Lemma [14, pp. 190–191, Theorem 14-7]. If $\lim_{n\to\infty} A_n = L$, then

$$\lim_{z\to 1^-} (1-z)\sum_{n=0}^{\infty} A_n z^n = L. \tag{7.3.3}$$

In the first step below, the insertion of z in the denominator introduces an extra expression that is equal to $-\alpha$ by Abel's Lemma. Thus, we must compensate for this insertion by adding α. Consequently, assuming that $\alpha_{-1} = 0$, we find that

$$\begin{aligned}
\lim_{z\to 1^-} \frac{d}{dz}(1-z)f(z) &= \lim_{z\to 1^-} \frac{d}{dz}(1-z)\frac{f(z)}{z} + \alpha \\
&= \lim_{z\to 1^-} \sum_{n=0}^{\infty} \left(n\alpha_n - (n-1)\alpha_{n-1} - \alpha_n\right) z^{n-2} + \alpha \\
&= \lim_{z\to 1^-} \sum_{n=0}^{\infty} \left(n\alpha_n - (n-1)\alpha_{n-1} - \alpha_n\right) z^{n} + \alpha \\
&= \lim_{z\to 1^-} \sum_{n=0}^{\infty} \left(-n(\alpha - \alpha_n) + (n-1)(\alpha - \alpha_{n-1})\right. \\
&\qquad\qquad \left. + (\alpha - \alpha_n)\right) z^n + \alpha \\
&= \lim_{z\to 1^-} \left(\sum_{n=0}^{\infty} -n(\alpha - \alpha_n) z^n + \sum_{n=-1}^{\infty} n(\alpha - \alpha_n) z^{n+1} \right) \\
&\quad + \lim_{z\to 1^-} \sum_{n=0}^{\infty} (\alpha - \alpha_n) z^n + \alpha \\
&= -\lim_{z\to 1^-} (1-z) \sum_{n=0}^{\infty} n(\alpha - \alpha_n) z^n - \alpha \\
&\quad + \lim_{z\to 1^-} \sum_{n=0}^{\infty} (\alpha - \alpha_n) z^n + \alpha
\end{aligned}$$

$$
\begin{aligned}
&= -\lim_{z\to 1^-}(1-z)\sum_{n=0}^{\infty} n(\alpha-\alpha_n)z^n + \lim_{z\to 1^-}\sum_{n=0}^{\infty}(\alpha-\alpha_n)z^n \\
&= 0 + \lim_{z\to 1^-}\sum_{n=0}^{\infty}(\alpha-\alpha_n)z^n \\
&= \sum_{n=0}^{\infty}(\alpha-\alpha_n),
\end{aligned}
$$

where in the penultimate line we applied (7.3.3) and (7.3.2), and where in the last line we used (7.3.1) and the fact that the series converges uniformly for $0 \le z \le 1$. ∎

We may now apply Theorem 7.3.1 to a couple of instances relevant to Ramanujan's work. The following theorem was first proved in [36, p. 397].

Theorem 7.3.2. *For $|a| < |t|$ and $a \neq 0$,*

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left(\frac{(t)_\infty}{(a)_\infty} - \frac{(t)_n}{(a)_n}\right) &= \sum_{n=1}^{\infty}\frac{(q/a)_n}{(q/t)_n}\left(\frac{a}{t}\right)^n \\
&+ \frac{(t)_\infty}{(a)_\infty}\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n} + \sum_{n=1}^{\infty}\frac{t^{-1}q^n}{1-t^{-1}q^n} - \sum_{n=0}^{\infty}\frac{tq^n}{1-tq^n} - \sum_{n=0}^{\infty}\frac{at^{-1}q^n}{1-at^{-1}q^n}\right).
\end{aligned}
$$

Proof. Using the q-binomial theorem (1.2.2), we can easily check that the conditions (7.3.1) and (7.3.2) of Theorem 7.3.1 are fulfilled for

$$\alpha_n := \frac{(t)_n}{(a)_n}.$$

After employing Theorem 7.3.1, we add and subtract terms in the sum on the right-hand side so that we can apply Ramanujan's ${}_1\psi_1$ summation (3.1.4). Then in the "added" sum, we make two applications of the easily verified identity

$$(c)_{-n} = \frac{(-1)^n q^{n(n+1)/2}}{c^n (q/c)_n}, \qquad n \ge 1.$$

In the third step below, we termwise differentiate the two series in powers of $1/z$ and simplify. Accordingly, for $|a/(tz)| < 1$, we find that

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left(\frac{(t)_\infty}{(a)_\infty} - \frac{(t)_n}{(a)_n}\right) &= \lim_{z\to 1^-}\frac{d}{dz}(1-z)\sum_{n=0}^{\infty}\frac{(t)_n}{(a)_n}z^n \\
&= \lim_{z\to 1^-}\frac{d}{dz}(1-z)\left(\sum_{n=-\infty}^{\infty}\frac{(t)_n}{(a)_n}z^n - \sum_{n=1}^{\infty}\frac{(q/a)_n}{(q/t)_n}\left(\frac{a}{tz}\right)^n\right) \\
&= \lim_{z\to 1^-}\frac{d}{dz}(1-z)\left(\frac{(q, a/t, tz, q/(tz); q)_\infty}{(a, q/t, z, a/(tz); q)_\infty}\right) + \sum_{n=1}^{\infty}\frac{(q/a)_n}{(q/t)_n}\left(\frac{a}{t}\right)^n
\end{aligned}
$$

$$= \frac{(q)_\infty (a/t)_\infty}{(a)_\infty (q/t)_\infty} \left(\frac{d}{dz} \frac{(tz)_\infty (q/(tz))_\infty}{(zq)_\infty (a/(tz))_\infty} \right)\Bigg|_{z=1} + \sum_{n=1}^{\infty} \frac{(q/a)_n}{(q/t)_n} \left(\frac{a}{t}\right)^n$$

$$= \frac{(t)_\infty}{(a)_\infty} \left(\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} + \sum_{n=1}^{\infty} \frac{t^{-1}q^n}{1-t^{-1}q^n} - \sum_{n=0}^{\infty} \frac{tq^n}{1-tq^n} - \sum_{n=0}^{\infty} \frac{at^{-1}q^n}{1-at^{-1}q^n} \right)$$

$$+ \sum_{n=1}^{\infty} \frac{(q/a)_n}{(q/t)_n} \left(\frac{a}{t}\right)^n,$$

as desired. □

Theorem 7.3.3. *For* $0 < |b| < 1$,

$$\sum_{n=0}^{\infty} \left(\frac{(a)_\infty (b)_\infty}{(q)_\infty (c)_\infty} - \frac{(a)_n (b)_n}{(q)_n (c)_n} \right)$$

$$= \frac{(a)_\infty (b)_\infty}{(q)_\infty (c)_\infty} \left(\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} - \sum_{n=0}^{\infty} \frac{aq^n}{1-aq^n} - \sum_{n=1}^{\infty} \frac{(c/b)_n b^n}{(a)_n (1-q^n)} \right).$$

Proof. We apply Theorem 7.3.1 when

$$\alpha_n = \frac{(a)_n (b)_n}{(q)_n (c)_n}. \tag{7.3.4}$$

We must check that the conditions (7.3.1) and (7.3.2) hold. First, we check that (7.3.1) holds. For brevity below, define

$$K(a,b,c,q) := \frac{(-|a|;|q|)_\infty (-|b|;|q|)_\infty}{(|q|;|q|)_\infty (|c|;|q|)_\infty}.$$

Hence, using the definition above and the q-binomial theorem (1.2.2) twice, we find that

$$\sum_{n=0}^{\infty} |\alpha - \alpha_n| = \sum_{n=0}^{\infty} \left| \frac{(a)_\infty (b)_\infty}{(q)_\infty (c)_\infty} - \frac{(a)_n (b)_n}{(q)_n (c)_n} \right|$$

$$\leq \frac{(-|a|;|q|)_\infty (-|b|;|q|)_\infty}{(|q|;|q|)_\infty (|c|;|q|)_\infty} \sum_{n=0}^{\infty} \left| 1 - \frac{(q^{n+1})_\infty (cq^n)_\infty}{(aq^n)_\infty (bq^n)_\infty} \right|$$

$$= K(a,b,c,q) \sum_{n=0}^{\infty} \left| 1 - \sum_{r=0}^{\infty} \frac{(q/a)_r a^r q^{nr}}{(q)_r} \sum_{s=0}^{\infty} \frac{(c/b)_s b^s q^{ns}}{(q)_s} \right|$$

$$= K(a,b,c,q) \sum_{n=0}^{\infty} \left| \sum_{\substack{r,s=0 \\ (r,s)\neq(0,0)}}^{\infty} \frac{(q/a)_r (c/b)_s a^r b^s q^{n(r+s)}}{(q)_r (q)_s} \right|$$

$$\leq K(a,b,c,q) \sum_{n=0}^{\infty} |q|^n \sum_{\substack{r,s=0 \\ (r,s)\neq(0,0)}}^{\infty} \frac{(-|q/a|;|q|)_r (-|c/b|;|q|)_s |a|^r |b|^s}{(|q|;|q|)_r (|q|;|q|)_s}$$

$$= \frac{K(a,b,c,q)}{1-|q|} \sum_{\substack{r,s=0\\(r,s)\neq(0,0)}}^{\infty} \frac{(-|q/a|;|q|)_r(-|c/b|;|q|)_s|a|^r|b|^s}{(|q|;|q|)_r(|q|;|q|)_s} < \infty,$$

where in the penultimate line we used the fact that $|q|^{n(r+s-1)} \leq 1$, since $r+s \geq 1$. This then proves that condition (7.3.1) is satisfied.

To prove (7.3.2), we first note that if $|q| < 1$, then $\lim_{n\to\infty} nq^n = 0$. Hence,

$$\lim_{n\to\infty} n(\alpha - \alpha_n) = \lim_{n\to\infty} nq^n \frac{(a)_\infty(b)_\infty}{(q)_\infty(c)_\infty} \times \left(\sum_{\substack{r,s=0\\(r,s)\neq(0,0)}}^{\infty} \frac{(q/a)_r(c/b)_s a^r b^s q^{n(r+s-1)}}{(q)_r(q)_s} \right).$$

Now, the expression inside parentheses above is bounded in absolute value, since we showed in our proof above that condition (7.3.1) holds. Since $\lim_{n\to\infty} nq^n = 0$, we see that we have shown that condition (7.3.2) is valid.

Hence, applying Theorem 7.3.1 with the value of α_n given in (7.3.4) and using Heine's transformation (1.2.1) with $h = 1$, we find that

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\frac{(a)_\infty(b)_\infty}{(q)_\infty(c)_\infty} - \frac{(a)_n(b)_n}{(q)_n(c)_n}\right) &= \lim_{z\to1^-} \frac{d}{dz}(1-z)\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(q)_n(c)_n}z^n \\ &= \lim_{z\to1^-}\frac{d}{dz}\frac{(b)_\infty(az)_\infty}{(c)_\infty(qz)_\infty}\left(1+\sum_{n=1}^{\infty}\frac{(c/b)_n(z)_n b^n}{(q)_n(az)_n}\right) \\ &= \frac{(b)_\infty(a)_\infty}{(c)_\infty(q)_\infty}\left(\sum_{n=1}^{\infty}\frac{q^n}{1-q^n} - \sum_{n=0}^{\infty}\frac{aq^n}{1-aq^n}\right) \\ &\quad - \frac{(b)_\infty(a)_\infty}{(c)_\infty(q)_\infty}\sum_{n=1}^{\infty}\frac{(c/b)_n b^n}{(a)_n(1-q^n)},\end{aligned}$$

as desired. □

We now come to three entries in the middle of page 14 of the lost notebook. The first of these is perhaps more appropriate for Chapter 1; however, it would be unfortunate to separate it from its central relationship to the remaining two entries in this section.

Entry 7.3.1 (p. 14). *We have*

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(-q)_n} = 1 + q\sum_{n=0}^{\infty}(-1)^n(q)_n q^n. \tag{7.3.5}$$

The function defined in this entry is quite remarkable. The coefficient of q^n is equal to the difference of the number of partitions of n into distinct parts with even rank and the number of partitions of n into distinct parts with odd rank. In [32], we further discuss this function when we examine Ramanujan's contributions to ranks and cranks in his lost notebook. It has been shown [34] that almost all of the coefficients in the power series expansion of this function are equal to 0; however, every integer appears infinitely often as a coefficient in the power series. A charming introduction to such phenomena has been provided by F.J. Dyson [148].

Proof. Applying (1.2.9) with $a = -q/\tau$, $b = q$, $c = xq$, and $t = \tau$, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(xq)_n} &= \lim_{\tau\to 0} \sum_{n=0}^{\infty} \frac{(-q/\tau)_n (q)_n \tau^n}{(q)_n (xq)_n} \\ &= (1-x) \sum_{n=0}^{\infty} (-q/x)_n x^n \\ &= 1 + \sum_{n=1}^{\infty} (-q/x)_n x^n - \sum_{n=1}^{\infty} (-q/x)_{n-1} x^n \\ &= 1 + \sum_{n=1}^{\infty} (-q/x)_{n-1} x^n q^n / x \\ &= 1 + \sum_{n=0}^{\infty} (-q/x)_n x^n q^{n+1}.\end{aligned}$$

If we now set $x = -1$, the desired result follows. □

Entry 7.3.2 (p. 14). *If* $S := (-q)_\infty$, *then*

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q)_n} = 2\left(\frac{S}{2} + \sum_{n=0}^{\infty} (S - (-q)_n) - S \sum_{n=1}^{\infty} \frac{q^n}{1-q^n}\right).$$

Proof. In Theorem 7.3.2, set $t = -1$ and let a tend to 0 to deduce that

$$\begin{aligned}&2(-q)_\infty - 1 + 2\sum_{n=1}^{\infty} ((-q)_\infty - (-q)_{n-1}) \\ &\qquad = \sum_{n=1}^{\infty} \frac{q^{n(n+1)/2}}{(-q)_n} + 2(-q)_\infty \left(\sum_{n=1}^{\infty} \frac{q^n}{1-q^n} + \frac{1}{2}\right),\end{aligned}$$

or, equivalently,

$$S + 2\sum_{n=0}^{\infty} (S - (-q)_n) - 2S \sum_{n=1}^{\infty} \frac{q^n}{1-q^n} = \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(-q)_n},$$

as desired. □

Entry 7.3.3 (p. 14). *Let* $S := 1/(q;q^2)_\infty$. *Then*

$$\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(-q;q)_n}=2\left(\frac{S}{2}+\sum_{n=0}^{\infty}\left(S-\frac{1}{(q;q^2)_{n+1}}\right)\right)-2S\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}.$$

Proof. In Theorem 7.3.3, replace q by q^2, then set $a=q^2$, $b=0$, and $c=q^3$, and lastly multiply both sides by $1/(1-q)$. Consequently,

$$\begin{aligned}\sum_{n=0}^{\infty}\left(S-\frac{1}{(q;q^2)_{n+1}}\right)&=\frac{1}{(q;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2+2n}}{(q^2;q^2)_n(1-q^{2n})}\\&=\frac{1}{(q;q^2)_\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2+2n}\left((1-q^{-n})+q^{-n}\right)}{(q^2;q^2)_n(1-q^{2n})}\\&=\frac{1}{(q;q^2)_\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^nq^{n^2+n}}{(q^2;q^2)_n(1+q^n)}\right.\\&\qquad\left.+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n^2+n}}{(q^2;q^2)_n(1-q^{2n})}\right)\\&=\frac{1}{2(q;q^2)_\infty}\lim_{\tau\to0}{}_3\phi_2\left(\begin{matrix}q/\tau,\ q/\tau,\ -1\\ -q,\ -q\end{matrix};q,-\tau^2\right)\\&\quad+\frac{1}{(q;q^2)_\infty}\left(-\frac{1}{2}+\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\right),\end{aligned}$$

which can be deduced from (1.3.1) or which can be found in N.J. Fine's book [149, p. 14, Equation (12.42)]. Providing the details for the former derivation, in (1.3.1), we replace q by q^2, set $a=z$ and $c=zq^2$, and let $b\to\infty$ to find that

$$\sum_{n=1}^{\infty}\frac{(-1)^n(z;q^2)_nq^{n(n+1)}}{(q^2;q^2)_n(zq^2;q^2)_n}=\frac{(q^2;q^2)_\infty}{(zq^2;q^2)_\infty}. \tag{7.3.6}$$

Differentiate both sides of (7.3.6) with respect to z and set $z=1$ to deduce that

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}q^{n(n+1)}}{(q^2;q^2)_n(1-q^{2n})}=\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}.$$

Next, we apply (2.1.1) with $a=b=q/\tau$, $c=-1$, and $d=e=-q$ to the far right side above and deduce that

$$\begin{aligned}\sum_{n=0}^{\infty}\left(S-\frac{1}{(q;q^2)_{n+1}}\right)&=\frac{1}{2(q;q^2)_\infty}\lim_{\tau\to0}\frac{1}{(-q;q)_\infty}{}_3\phi_2\left(\begin{matrix}q/\tau,\ -\tau,\ q\\ -q,\ -q\tau\end{matrix};q,-\tau\right)\\&\quad+\frac{1}{(q;q^2)_\infty}\left(-\frac{1}{2}+\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\right)\end{aligned}$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(-q;q)_n} + \frac{1}{(q;q^2)_\infty}\left(-\frac{1}{2}+\sum_{n=1}^{\infty}\frac{q^{2n}}{1-q^{2n}}\right).$$

Multiplying throughout by 2, we obtain a result equivalent to the asserted identity. $\square$

Theorems 7.3.2 and 7.3.3 and Entries 7.3.2 and 7.3.3 are also discussed in Ono's monograph [224, pp. 168–174]. Further identities arising from Theorems 7.3.2 and 7.3.3 can be found in [224]. R. Chapman [121] has devised combinatorial proofs of several of these identities. W.Y.C. Chen and K.Q. Ji [123] have devised beautiful combinatorial proofs of the last two entries.

7.4 Innocents Abroad

In order to prove the two results listed in Entry 7.4.1 below, we must collect a number of esoteric results. Of these, the three identities for very well-poised ${}_{10}\phi_9$'s seem especially as though they ought to be unnecessary.

Lemma 7.4.1. *For each nonnegative integer m,*

$$(q;q^2)_m\sum_{n=0}^{2m}\frac{(-1)^n}{(\alpha;q)_n(q;q)_{2m-n}} = \sum_{n=0}^{m}\frac{(-1)^n q^{n^2}}{(q^2;q^2)_{m-n}(\alpha q;q^2)_n}. \tag{7.4.1}$$

The case $\alpha = 0$ appears as Lemma 6.5.2 in Chapter 6. The case $\alpha = q$ reduces easily upon the use of (1.7.3) to a famous formula of Gauss for the Gaussian polynomials. This formula was used by Gauss [152] in his famous elementary evaluation of the quadratic Gauss sum. For further details, see the book of either H. Rademacher [232, pp. 85–88] or Berndt, R.J. Evans, and K.S. Williams [72, pp. 20–24].

Proof. We set

$$L_m(\alpha) := (q;q^2)_m\sum_{n=0}^{2m}\frac{(-1)^n}{(\alpha;q)_n(q;q)_{2m-n}}, \tag{7.4.2}$$

$$R_m(\alpha) := \sum_{n=0}^{m}\frac{(-1)^n q^{n^2}}{(q^2;q^2)_{m-n}(\alpha q;q^2)_n}. \tag{7.4.3}$$

Our lemma follows immediately by mathematical induction once we prove that each of $L_m(\alpha)$ and $R_m(\alpha)$ satisfies the following initial conditions and recurrences:

$$f_0(\alpha) = 1, \tag{7.4.4}$$

$$(1-\alpha)f_{m+1}(\alpha) - \frac{1-q^{2m+1}}{1-\alpha q}f_m(\alpha q^2) = \frac{q^{2m+2}-\alpha}{(q^2;q^2)_{m+1}}. \tag{7.4.5}$$

Clearly, $R_0(\alpha) = L_0(\alpha) = 1$; so (7.4.4) holds for each of $L_m(\alpha)$ and $R_m(\alpha)$.

Next,

$$\begin{aligned}
&L_{m+1}(\alpha) - \frac{1-q^{2m+1}}{(1-\alpha)(1-\alpha q)} L_m(\alpha q^2) \\
&= (q;q^2)_{m+1} \left(\sum_{n=0}^{2m+2} \frac{(-1)^n}{(\alpha;q)_n (q;q)_{2m-n+2}} - \sum_{n=0}^{2m} \frac{(-1)^n}{(\alpha;q)_{n+2}(q;q)_{2m-n}} \right) \\
&= (q;q^2)_{m+1} \left(\sum_{n=-2}^{2m} \frac{(-1)^n}{(\alpha;q)_{n+2}(q;q)_{2m-n}} - \sum_{n=0}^{2m} \frac{(-1)^n}{(\alpha;q)_{n+2}(q;q)_{2m-n}} \right) \\
&= (q;q^2)_{m+1} \left(\frac{1}{(q;q)_{2m+2}} - \frac{1}{(1-\alpha)(q;q)_{2m+1}} \right) \\
&= \frac{(q;q^2)_{m+1}}{(1-\alpha)(q;q)_{2m+2}} \left((1-\alpha) - (1-q^{2m+2}) \right) \\
&= \frac{q^{2m+2}-\alpha}{(1-\alpha)(q^2;q^2)_{m+1}}.
\end{aligned}$$

Thus, (7.4.5) holds for $L_m(\alpha)$.

Finally,

$$\begin{aligned}
&(1-\alpha)R_{m+1}(\alpha) - \frac{R_m(\alpha q^2)}{1-\alpha q} \\
&\quad = \sum_{n=0}^{m+1} \frac{(-1)^n q^{n^2}}{(q^2;q^2)_{m+1-n}(\alpha q;q^2)_n} \left(1 - \alpha - \frac{1-q^{2m+2-2n}}{1-\alpha q^{2n+1}} \right) \\
&\quad = \sum_{n=0}^{m+1} \frac{(-1)^n q^{n^2} (q^{2m+2-2n} - \alpha q^{2n+1} + \alpha^2 q^{2n+1} - \alpha)}{(q^2;q^2)_{m+1-n}(\alpha q;q^2)_{n+1}} \\
&\quad = \frac{q^{2m+2}-\alpha}{(q^2;q^2)_{m+1}} - \frac{\alpha q}{(1-\alpha q)(q^2;q^2)_m} \\
&\qquad + \sum_{n=1}^{m+1} \frac{(-1)^n q^{n^2} (q^{2m+2-2n} - \alpha q^{2n+1} + \alpha^2 q^{2n+1} - \alpha)}{(q^2;q^2)_{m+1-n}(\alpha q;q^2)_{n+1}} \\
&\quad = \frac{q^{2m+2}-\alpha}{(q^2;q^2)_{m+1}} - \frac{\alpha q}{(1-\alpha q)(q^2;q^2)_m} \\
&\qquad + \sum_{n=1}^{m+1} \frac{(-1)^n q^{n^2} \left(q^{2m+2-2n}(1-\alpha q^{2n+1}) \right)}{(q^2;q^2)_{m+1-n}(\alpha q;q^2)_{n+1}} \\
&\qquad + \sum_{n=1}^{m+1} \frac{(-1)^n q^{n^2} (\alpha q^{2m+3} - \alpha q^{2n+1} + \alpha^2 q^{2n+1} - \alpha)}{(q^2;q^2)_{m+1-n}(\alpha q;q^2)_{n+1}} \\
&\quad = \frac{q^{2m+2}-\alpha}{(q^2;q^2)_{m+1}} - \frac{q^{2m+1} R_m(\alpha q^2)}{1-\alpha q} + \Bigg\{ - \frac{\alpha q}{(1-\alpha q)(q^2;q^2)_m}
\end{aligned}$$

$$+\sum_{n=1}^{m+1}\frac{(-1)^n q^{n^2}(\alpha q^{2m+3}-\alpha q^{2n+1}+\alpha^2 q^{2n+1}-\alpha)}{(q^2;q^2)_{m+1-n}(\alpha q;q^2)_{n+1}}\Bigg\}.$$

Consequently, to show that $R_n(\alpha)$ satisfies (7.4.5), we need only show that the expression above inside the curly brackets is identically equal to 0. To that end,

$$\begin{aligned}
&-\frac{\alpha q}{(1-\alpha q)(q^2;q^2)_m}+\sum_{n=1}^{m+1}\frac{(-1)^n q^{n^2}(\alpha q^{2m+3}-\alpha q^{2n+1}+\alpha^2 q^{2n+1}-\alpha)}{(q^2;q^2)_{m+1-n}(\alpha q;q^2)_{n+1}}\\
&\quad=-\frac{\alpha q}{(1-\alpha q)(q^2;q^2)_m}\\
&\qquad+\sum_{n=1}^{m+1}\frac{(-1)^n q^{n^2}(-\alpha q^{2n+1}(1-q^{2m-2n+2})-\alpha(1-\alpha q^{2n+1}))}{(q^2;q^2)_{m+1-n}(\alpha q;q^2)_{n+1}}\\
&\quad=-\alpha\sum_{n=0}^{m}\frac{(-1)^n q^{(n+1)^2}}{(q^2;q^2)_{m-n}(\alpha q;q^2)_{n+1}}-\alpha\sum_{n=1}^{m+1}\frac{(-1)^n q^{n^2}}{(q^2;q^2)_{m+1-n}(\alpha q;q^2)_n}\\
&\quad=-\alpha\sum_{n=0}^{m}\frac{(-1)^n q^{(n+1)^2}}{(q^2;q^2)_{m-n}(\alpha q;q^2)_{n+1}}+\alpha\sum_{n=0}^{m}\frac{(-1)^n q^{(n+1)^2}}{(q^2;q^2)_{m-n}(\alpha q;q^2)_{n+1}}\\
&\quad=0.
\end{aligned}$$

□

Lemma 7.4.2. *For each positive integer m,*

$$(q;q^2)_m\sum_{n=0}^{2m-1}\frac{(-1)^n}{(q;q)_n(\alpha;q)_{2m-1-n}}=\left(1-\frac{\alpha}{q}\right)\sum_{n=1}^{m}\frac{(-1)^n q^{n^2}}{(q^2;q^2)_{m-n}(\alpha;q^2)_n}.$$

Proof. Following in the wake of Lemma 7.4.1, we see that

$$\begin{aligned}
\left(1-\frac{\alpha}{q}\right)\sum_{n=0}^{2m-1}\frac{(-1)^n}{(q;q)_n(\alpha/q;q)_{2m-n}}&=\left(1-\frac{\alpha}{q}\right)\left(\frac{L_m(\alpha/q)}{(q;q^2)_m}-\frac{1}{(q;q)_{2m}}\right)\\
&=\left(1-\frac{\alpha}{q}\right)\left(\frac{R_m(\alpha/q)}{(q;q^2)_m}-\frac{1}{(q;q)_{2m}}\right)\\
&=\frac{1-\alpha/q}{(q;q^2)_m}\sum_{n=1}^{m}\frac{(-1)^n q^{n^2}}{(q^2;q^2)_{m-n}(\alpha;q^2)_n},
\end{aligned}$$

where we have used the definitions of $L_m(\alpha)$ and $R_m(\alpha)$, given in (7.4.2) and (7.4.3), respectively, and where we have appealed to Lemma 7.4.1. This completes the proof. □

In addition to these two lemmas, we require three identities for the function ${}_{10}\phi_9$ (see (2.1.2) for this notation). The first two were given by W.N. Bailey [41, equations (6.1), (6.3)] and are given by

$$\lim_{N\to\infty} {}_{10}\phi_9\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, r_1, -r_1, r_2, -r_2, q^{-N}, -q^{-N} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{r_1}, -\frac{aq}{r_1}, \frac{aq}{r_2}, -\frac{aq}{r_2}, aq^{N+1}, -aq^{N+1}\end{matrix}; q, -\frac{a^3q^{2N+3}}{br_1^2r_2^2}\right)$$
$$= \frac{(a^2q^2;q^2)_\infty(a^2q^2/(r_1^2r_2^2);q^2)_\infty}{(a^2q^2/r_1^2;q^2)_\infty(a^2q^2/r_2^2;q^2)_\infty}$$
$$\times \sum_{n=0}^{\infty} \frac{(r_1^2;q^2)_n(r_2^2;q^2)_n(-aq/b;q)_{2n}}{(q^2;q^2)_n(a^2q^2/b^2;q^2)_n(-aq;q)_{2n}}\left(\frac{a^2q^2}{r_1^2r_2^2}\right)^n \tag{7.4.6}$$

and

$$\lim_{N\to\infty} {}_{10}\phi_9\left(\begin{matrix} a, q^2\sqrt{a}, -q^2\sqrt{a}, p_1, p_1q, p_2, p_2q, f, q^{-2N}, q^{-2N+1} \\ \sqrt{a}, -\sqrt{a}, \frac{aq^2}{p_1}, \frac{aq}{p_1}, \frac{aq^2}{p_2}, \frac{aq}{p_2}, \frac{aq^2}{f}, aq^{2N+2}, aq^{2N+1}\end{matrix}; q^2, \frac{a^3q^{4N+3}}{p_1^2p_2^2f}\right)$$
$$= \frac{(aq;q)_\infty(aq/(p_1p_2);q)_\infty}{(aq/p_1;q)_\infty(aq/p_2;q)_\infty}\sum_{n=0}^{\infty}\frac{(p_1;q)_n(p_2;q)_n(aq/f;q^2)_n}{(q;q)_n(aq;q^2)_n(aq/f;q)_n}\left(\frac{aq}{p_1p_2}\right)^n. \tag{7.4.7}$$

The third is the case $k=3$ of a generalization [17, Theorem 4] of equation (12.2.1) of our first book [31, p. 262], namely,

$${}_{10}\phi_9\left(\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b_1, c_1, b_2, c_2, b_3, c_3, q^{-N} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b_1}, \frac{aq}{c_1}, \frac{aq}{b_2}, \frac{aq}{c_2}, \frac{aq}{b_3}, \frac{aq}{c_3}, aq^{N+1}\end{matrix}; q, \frac{a^3q^{N+3}}{b_1b_2b_3c_1c_2c_3}\right)$$
$$= \frac{(aq;q)_N(aq/(b_3c_3);q)_N}{(aq/b_3;q)_N(aq/c_3;q)_N} \tag{7.4.8}$$
$$\times \sum_{m_1,m_2=0}^{\infty}\frac{(aq/(b_1c_1);q)_{m_1}(aq/(b_2c_2);q)_{m_2}(b_2;q)_{m_1}(c_2;q)_{m_1}}{(q;q)_{m_1}(q;q)_{m_2}(aq/b_1;q)_{m_1}(aq/c_1;q)_{m_1}}$$
$$\times \frac{(b_3;q)_{m_1+m_2}(c_3;q)_{m_1+m_2}(q^{-N};q)_{m_1+m_2}(aq)^{m_1}q^{m_1+m_2}}{(aq/b_2;q)_{m_1+m_2}(aq/c_2;q)_{m_1+m_2}(b_3c_3q^{-N}/a;q)_{m_1+m_2}(b_2c_2)^{m_1}}.$$

Entry 7.4.1 (p. 25). *We have*

$$\sum_{n=0}^{\infty}\frac{(-1)^nq^{n(n+1)/2}}{(-q;q)_n^2} = \sum_{n=0}^{\infty}\frac{(-q)^{n(n+1)/2}}{(-q^2;q^2)_n} - 2\sum_{n=1}^{\infty}\frac{(-1)^nq^{2n^2}}{(-q;q^2)_{2n}}, \tag{7.4.9}$$
$$\varphi(-q)\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^2;q^2)_n} = \sum_{n=0}^{\infty}\frac{(-q)^{n(n+1)/2}}{(-q^2;q^2)_n} + 2\sum_{n=1}^{\infty}\frac{(-1)^nq^{2n^2}}{(-q;q^2)_{2n}}. \tag{7.4.10}$$

We follow closely the proof in [23]. Unfortunately, this is currently the only known proof.

Proof. Instead of proving each of these formulas independently, we prove the following two formulas:

$$\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^2;q^2)_n}+\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n^2}=2\sum_{n=0}^{\infty}\frac{(-q)^{n(n+1)/2}}{(-q^2;q^2)_n}, \tag{7.4.11}$$

$$\frac{(q;q)_\infty}{(-q;q)_\infty}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^2;q^2)_n}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n^2}=4\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n^2}}{(-q;q^2)_{2n}}. \tag{7.4.12}$$

To obtain (7.4.9) from the latter two equalities, merely subtract (7.4.12) from (7.4.11) and divide both sides by 2. To obtain (7.4.10), add (7.4.11) and (7.4.12), recall (1.4.9), and divide the result by 2.

We now prove (7.4.11). Using (1.2.4) twice, setting $N = m + n$, invoking Lemma 7.4.1, inverting the order of summation on N and j below, and replacing N by $N + j$, we may write the left-hand side of (7.4.11) as

$$\begin{aligned}
&\frac{1}{(-q;q)_\infty}\left\{\sum_{m=0}^{\infty}\frac{q^{m(m+1)/2}}{(-q;q)_m}(q^{m+1};q)_\infty+\sum_{m=0}^{\infty}\frac{(-1)^m q^{m(m+1)/2}}{(-q;q)_m}(-q^{m+1};q)_\infty\right\}\\
&=\frac{1}{(-q;q)_\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{q^{m(m+1)/2+n(n+1)/2+mn}\left((-1)^n+(-1)^m\right)}{(-q;q)_m(q;q)_n}\\
&=\frac{1}{(-q;q)_\infty}\sum_{N=0}^{\infty}q^{N(N+1)/2}\sum_{n=0}^{N}\frac{\left((-1)^n+(-1)^{N-n}\right)}{(-q;q)_{N-n}(q;q)_n}\\
&=\frac{2}{(-q;q)_\infty}\sum_{N=0}^{\infty}q^{N(2N+1)}\sum_{n=0}^{2N}\frac{(-1)^n}{(-q;q)_{2N-n}(q;q)_n}\\
&=\frac{2}{(-q;q)_\infty}\sum_{N=0}^{\infty}\frac{q^{N(2N+1)}}{(q;q^2)_N}\sum_{j=0}^{N}\frac{(-1)^j q^{j^2}}{(q^2;q^2)_{N-j}(-q^2;q^2)_j}\\
&=\frac{2}{(-q;q)_\infty}\sum_{j=0}^{\infty}\sum_{N=0}^{\infty}\frac{(-1)^j q^{(N+j)(2N+2j+1)}q^{j^2}}{(-q^2;q^2)_j(q;q^2)_{N+j}(q^2;q^2)_N}\\
&=\frac{2}{(-q;q)_\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q^{3j^2+j}}{(-q^2;q^2)_j(q;q^2)_j}\sum_{N=0}^{\infty}\frac{q^{2N^2+N+4Nj}}{(q^2;q^2)_N(q^{2j+1};q^2)_N}\\
&=:\frac{2}{(-q;q)_\infty}\sum_{j=0}^{\infty}\frac{(-1)^j q^{3j^2+j}}{(-q^2;q^2)_j(q;q^2)_j}S_1(j).
\end{aligned}\tag{7.4.13}$$

We now transform $S_1(j)$ by rewriting it as a limit of a certain ${}_2\phi_1$ and using the second Heine transformation, Corollary 1.2.4, to find that

$$\begin{aligned}
S_1(j)&=\lim_{\tau\to 0}{}_2\phi_1\left(\begin{matrix}q/\tau,q/\tau\\ q^{2j+1}\end{matrix};q^2,\tau^2q^{4j+1}\right)\\
&=\lim_{\tau\to 0}\frac{(\tau q^{2j};q^2)_\infty(\tau q^{4j+2};q^2)_\infty}{(q^{2j+1};q^2)_\infty(\tau^2q^{4j+1};q^2)_\infty}{}_2\phi_1\left(\begin{matrix}q^{2j+2},q/\tau\\ \tau q^{4j+2}\end{matrix};q^2,\tau q^{2j}\right)
\end{aligned}$$

$$= \frac{1}{(q^{2j+1};q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n (q^{2j+2};q^2)_n q^{n^2+2jn}}{(q^2;q^2)_n}. \tag{7.4.14}$$

Putting (7.4.14) in (7.4.13), simplifying with the use of Euler's identity, and lastly using (7.4.8), we deduce that

$$\begin{aligned}
&\frac{1}{(-q;q)_\infty}\left\{\sum_{m=0}^{\infty}\frac{q^{m(m+1)/2}}{(-q;q)_m}(q^{m+1};q)_\infty+\sum_{m=0}^{\infty}\frac{(-1)^m q^{m(m+1)/2}}{(-q;q)_m}(-q^{m+1};q)_\infty\right\}\\
&=2\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{n+j}(q^2;q^2)_{n+j}q^{(n+j)^2+2j^2+j}}{(q^2;q^2)_j(q^2;q^2)_n(-q^2;q^2)_j}\\
&=2(1-q)\lim_{c_1,b_2,c_2,b_3,N\to\infty}\\
&\quad {}_{10}\phi_9\left(\begin{matrix} q,q^{5/2},-q^{5/2},-q,c_1,b_2,c_2,b_3,q^2,q^{-2N} \\ q^{1/2},-q^{1/2},-q^2,\dfrac{q^3}{c_1},\dfrac{q^3}{b_2},\dfrac{q^3}{c_2},\dfrac{q^3}{b_3},q,q^{2N+3}\end{matrix};q^2,-\frac{q^{2N+6}}{c_1b_2c_2b_3}\right)\\
&=2(1-q)\lim_{f,r_2,N\to\infty} \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (7.4.15)\\
&\quad {}_{10}\phi_9\left(\begin{matrix} q,q^{5/2},-q^{5/2},-q,q^2,f,r_2,-r_2q,q^{-2N},-q^{-2N+1} \\ q^{1/2},-q^{1/2},-q^2,q,\dfrac{q^3}{f},\dfrac{q^3}{r_2},-\dfrac{q^2}{r_2},q^{2N+3},-q^{2N+2}\end{matrix};q^2,-\frac{q^{4N+4}}{fr_2^2}\right),
\end{aligned}$$

where we set $c_1 = f$, $b_2 = r_2$, $c_2 = -r_2q$, and $b_3 = -q^{-2N+1}$. Observe that each ${}_{10}\phi_9$ has five upper-row parameters tending to ∞ and five lower-row parameters tending to 0, while the other parameters in both upper and lower rows remain unchanged. Next, in (7.4.7), set $p_1 = q$ and let p_2, f, and N tend to ∞. Hence,

$$\begin{aligned}
&\lim_{p_2,f,N\to\infty} {}_{10}\phi_9\\
&\times\left(\begin{matrix} a,q^2\sqrt{a},-q^2\sqrt{a},q,q^2,p_2,p_2q,f,q^{-2N},q^{-2N+1} \\ \sqrt{a},-\sqrt{a},aq,a,\dfrac{aq^2}{p_2},\dfrac{aq}{p_2},\dfrac{aq^2}{f},aq^{2N+2},aq^{2N+1}\end{matrix};q^2,\frac{a^3q^{4N+1}}{fp_2^2}\right)\\
&=\frac{1}{1-a}\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n-1)/2}}{(aq;q^2)_n}.
\end{aligned}\tag{7.4.16}$$

Replace q by $-q$ in (7.4.16) and then set $a = q$. Lastly, replace the identity that we thus obtain in (7.4.15) to conclude that

$$\begin{aligned}
&\frac{1}{(-q;q)_\infty}\left\{\sum_{m=0}^{\infty}\frac{q^{m(m+1)/2}}{(-q;q)_m}(q^{m+1};q)_\infty+\sum_{m=0}^{\infty}\frac{(-1)^m q^{m(m+1)/2}}{(-q;q)_m}(-q^{m+1};q)_\infty\right\}\\
&=2\sum_{n=0}^{\infty}\frac{(-q)^{n(n+1)/2}}{(-q^2;q^2)_n}.
\end{aligned}$$

This then finally concludes the proof of (7.4.11).

To complete the proof of Entry 7.4.1, we now prove (7.4.12). The steps are like those in the proof of (7.4.11). Using (1.2.4) twice, setting $N = m+n$, invoking Lemma 7.4.2 with $\alpha = -q$, and inverting the order of summation on N and j below, we may write the left-hand side of (7.4.12) as

$$\begin{aligned}
&\frac{1}{(-q;q)_\infty}\left\{\sum_{m=0}^\infty \frac{q^{m(m+1)/2}}{(-q;q)_m}(q^{m+1};q)_\infty - \sum_{m=0}^\infty \frac{(-1)^m q^{m(m+1)/2}}{(-q;q)_m}(-q^{m+1};q)_\infty\right\}\\
&\qquad= \frac{1}{(-q;q)_\infty}\sum_{m=0}^\infty\sum_{n=0}^\infty \frac{q^{m(m+1)/2+n(n+1)/2+mn}\left((-1)^n-(-1)^m\right)}{(-q;q)_m(q;q)_n}\\
&\qquad= \frac{1}{(-q;q)_\infty}\sum_{N=0}^\infty q^{N(N+1)/2}\sum_{n=0}^N \frac{\left((-1)^n-(-1)^{N-n}\right)}{(-q;q)_{N-n}(q;q)_n}\\
&\qquad= \frac{2}{(-q;q)_\infty}\sum_{N=0}^\infty q^{(N+1)(2N+1)}\sum_{n=0}^{2N+1} \frac{(-1)^n}{(-q;q)_{2N+1-n}(q;q)_n}\\
&\qquad= \frac{4}{(-q;q)_\infty}\sum_{N=0}^\infty \frac{q^{(N+1)(2N+1)}}{(q;q^2)_{N+1}}\sum_{j=0}^N \frac{(-1)^{j+1}q^{(j+1)^2}}{(q^2;q^2)_{N-j}(-q;q^2)_{j+1}}\\
&\qquad= \frac{4}{(-q;q)_\infty}\sum_{j=0}^\infty\sum_{N=0}^\infty \frac{(-1)^{j+1}q^{(N+j+1)(2N+2j+1)}q^{(j+1)^2}}{(-q;q^2)_{j+1}(q;q^2)_{N+j+1}(q^2;q^2)_N}\\
&\qquad= \frac{4}{(-q;q)_\infty}\sum_{j=0}^\infty \frac{(-1)^{j+1}q^{3j^2+5j+2}}{(-q;q^2)_{j+1}(q;q^2)_{j+1}}\sum_{N=0}^\infty \frac{q^{2N^2+3N+4Nj}}{(q^2;q^2)_N(q^{2j+3};q^2)_N}\\
&\qquad=: \frac{4}{(-q;q)_\infty}\sum_{j=0}^\infty \frac{(-1)^{j+1}q^{3j^2+5j+2}}{(-q;q^2)_{j+1}(q;q^2)_{j+1}}S_2(j). \qquad (7.4.17)
\end{aligned}$$

As in the proof of (7.4.11), we transform $S_2(j)$ by rewriting it as a limit of a certain ${}_2\phi_1$ and using the second Heine transformation, Corollary 1.2.4, to find that

$$\begin{aligned}
S_2(j) &= \lim_{\tau\to 0} {}_2\phi_1\left(\begin{matrix} q/\tau, q/\tau \\ q^{2j+3}\end{matrix}; q^2, \tau^2 q^{4j+3}\right)\\
&= \lim_{\tau\to 0} \frac{(\tau q^{2j+2};q^2)_\infty(\tau q^{4j+4};q^2)_\infty}{(q^{2j+3};q^2)_\infty(\tau^2q^{4j+3};q^2)_\infty}\,{}_2\phi_1\left(\begin{matrix} q^{2j+2}, q/\tau \\ \tau q^{4j+4}\end{matrix}; q^2, \tau q^{2j+2}\right)\\
&= \frac{1}{(q^{2j+3};q^2)_\infty}\sum_{n=0}^\infty \frac{(-1)^n(q^{2j+2};q^2)_n q^{n^2+2jn+2n}}{(q^2;q^2)_n}. \qquad (7.4.18)
\end{aligned}$$

Putting (7.4.18) in (7.4.17), employing (7.4.8), and using Euler's identity, we find that

$$\frac{1}{(-q;q)_\infty}\left\{\sum_{m=0}^\infty \frac{q^{m(m+1)/2}}{(-q;q)_m}(q^{m+1};q)_\infty - \sum_{m=0}^\infty \frac{(-1)^m q^{m(m+1)/2}}{(-q;q)_m}(-q^{m+1};q)_\infty\right\}$$

$$
\begin{aligned}
&= 4\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(-1)^{j+n+1}(q^2;q^2)_{j+n}q^{(n+j)^2+2j^2+3j+2+2(j+n)}}{(-q;q^2)_{j+1}(q^2;q^2)_j(q^2;q^2)_n} \\
&= -\frac{4q^2(1-q^3)}{1+q}\lim_{c_1,b_2,c_2,b_3,N\to\infty} \\
&\quad {}_{10}\phi_9\left(\begin{matrix} q^3, q^{7/2}, -q^{7/2}, -q^2, c_1, b_2, c_2, b_3, q^2, q^{-2N} \\ q^{3/2}, -q^{3/2}, q^3, -q^3, \frac{q^5}{c_1}, \frac{q^5}{b_2}, \frac{q^5}{c_2}, \frac{q^5}{b_3}, q^{2N+5} \end{matrix}; q^2, -\frac{q^{2N+11}}{c_1b_2c_2b_3}\right) \\
&= -\frac{4q^2(1-q^3)}{1+q}\lim_{N,b,r_2\to\infty} \\
&\quad {}_{10}\phi_9\left(\begin{matrix} q^3, q^{7/2}, -q^{7/2}, b, q^2, -q^2, r_2, -r_2, q^{-2N}, -q^{-2N} \\ q^{3/2}, -q^{3/2}, q^3, -q^3, \frac{q^5}{r_2}, -\frac{q^5}{r_2}, \frac{q^5}{b}, q^{2N+5}, -q^{2N+5} \end{matrix}; q^2, -\frac{q^{4N+11}}{br_2^2}\right) \\
&= -\frac{4q^2(1-q^3)}{1+q}\frac{1}{1-q^6}\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+4n}}{(-q^5;q^2)_{2n}} \\
&= 4\sum_{n=0}^{\infty}\frac{(-1)^{n-1}q^{2(n+1)^2}}{(-q;q^2)_{2n+2}} \\
&= 4\sum_{n=1}^{\infty}\frac{(-1)^n q^{2n^2}}{(-q;q^2)_{2n}},
\end{aligned}
\tag{7.4.19}
$$

where in the antepenultimate line of (7.4.19), we applied (7.4.6) with q first replaced by q^2, then with $a = q^3$ and $r_1 = q^2$, and lastly with b, r_2, and N tending to ∞. This finally completes the proof of (7.4.12). □

8

Theta Function Identities

8.1 Introduction

Theta function identities are ubiquitous in Ramanujan's notebooks [243]; in particular, see Berndt's books [54] and [57] for several hundred such identities. Theta function identities are also prominent in Ramanujan's lost notebook [244]. Several of these identities are intimately connected with the Rogers–Ramanujan continued fraction, and so these were examined in Chapter 1 of [31]. However, other chapters in [31] contain theta function identities, with Chapters 4, 13, 15, and 17 being exceptionally fruitful sources. Readers having read the first seven chapters of the present volume have seen how theta functions make perhaps unexpected appearances in q-series identities. Readers who continue to read the remainder of this volume will observe that theta functions and their identities are inextricably intertwined with Eisenstein series. What is rewarding and refreshing about Ramanujan's identities involving theta functions is that he often discovers types of theta function identities that were previously unknown to us. His identities are also frequently surprising, both in their forms and in their appearances with other mathematical objects.

In this chapter, we offer some of these beautiful identities, most of which were first proved by S.H. Son [266], [267]. Indeed, we follow precisely Son's beautiful proofs in [266] and [267] in the first two sections. For the convenience of the reader, we begin by reviewing some relevant notation, definitions, and theorems of Ramanujan that we need.

Ramanujan's theta function $f(a,b)$ is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1.$$

Perhaps the most important property of $f(a,b)$ is the Jacobi triple product identity.

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Theorem 8.1.1 (Jacobi Triple Product Identity). *For* $|ab| < 1$,

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{8.1.1}$$

The three most important special cases of $f(a,b)$ are defined by, in Ramanujan's notation,

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2}, \tag{8.1.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \tag{8.1.3}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \tag{8.1.4}$$

where the last equality is Euler's pentagonal number theorem, an easy consequence of the Jacobi triple product identity.

We use the following three basic theorems [54, pp. 48, 34, 80] of Ramanujan. One of them is the quintuple product identity, which we previously recorded in (3.1.2) and in Entry 3.1.1, and which we restate for convenience.

Theorem 8.1.2. *Let* $U_n = a^{n(n+1)/2}b^{n(n-1)/2}$ *and* $V_n = a^{n(n-1)/2}b^{n(n+1)/2}$ *for each integer* n. *Then*

$$f(a,b) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right). \tag{8.1.5}$$

Theorem 8.1.3. *For* $|ab| < 1$ *and each nonnegative integer* n,

$$f(a,b) = a^{n(n+1)/2}b^{n(n-1)/2} f\big(a(ab)^n, b(ab)^{-n}\big). \tag{8.1.6}$$

Theorem 8.1.4 (Quintuple Product Identity). *For* $|\lambda x^3| < 1$,

$$f(-\lambda^2x^3, -\lambda x^6) + x f(-\lambda, -\lambda^2 x^9) = \frac{f(-x^2,-\lambda x) f(-\lambda x^3)}{f(-x,-\lambda x^2)}. \tag{8.1.7}$$

The following theorem of Son [265], [31, p. 14, Lemma 1.2.4] is also needed.

Theorem 8.1.5. *Let* $|ab| < 1$, *let* j *and* k *denote arbitrary integers, let* p *be a prime, let* $\zeta := \exp(2\pi i/p)$, *and let* $x = s$, $0 \le s < p$, *be a solution of*

$$(j+k)x + j \equiv 0 \pmod{p}.$$

Then

$$\prod_{n=1}^{p} f(-\zeta^{jn}a, -\zeta^{kn}b) \tag{8.1.8}$$

$$= \begin{cases} \dfrac{f^p(-a^{s+1}b^s, -a^{p-s-1}b^{p-s})f(-a^p, -b^p)}{f(-a^{p(s+1)}b^{ps}, -a^{p(p-s-1)}b^{p(p-s)})}, & \textit{if} \quad j+k \not\equiv 0 \pmod p, \\ f^p(-ab)\dfrac{f(-a^p, -b^p)}{f(-a^pb^p)}, & \textit{if} \quad j+k \equiv 0 \pmod p. \end{cases}$$

We also utilize an identity established to prove a formula in Ramanujan's notebooks [56, p. 143], [55].

Lemma 8.1.1.

$$f^3(a^3b^6, a^6b^3) + a^3f^3(b^3, a^9b^6) + b^3f^3(a^3, a^6b^9)$$
$$= \frac{f(a^3, b^3)f^3(-ab)}{f(-a^3b^3)} + 3ab\frac{f(a^3, b^3)f^3(-a^9b^9)}{f(-a^3b^3)}. \tag{8.1.9}$$

If $q = \exp(-\pi\sqrt{n})$, for some positive rational number n, the Ramanujan–Weber class invariant G_n is defined by

$$G_n := 2^{-1/4}q^{-1/24}(-q; q^2)_\infty. \tag{8.1.10}$$

In his notebooks [243], Ramanujan recorded many values of class invariants; see [57, Chapter 34] for proofs of most of these values.

Lastly, we record the famous Rogers–Ramanujan identities, which we have previously recorded in (4.1.1) and (4.1.2).

Theorem 8.1.6. *For* $|q| < 1$,

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q; q^5)_\infty (q^4; q^5)_\infty}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q; q)_n} = \frac{1}{(q^2; q^5)_\infty (q^3; q^5)_\infty}.$$

See Chapter 10 of [31] for Ramanujan's discussion (and our discussion of Ramanujan's discussion) of these identities in his lost notebook.

8.2 Cubic Identities

It will be convenient to define

$$F_0 : = f(a^3b^6, a^6b^3), \tag{8.2.1}$$
$$F_1 : = a\, f(b^3, a^9b^6), \tag{8.2.2}$$
$$F_2 : = b\, f(a^3, a^6b^9), \tag{8.2.3}$$

and

$$\omega : = \exp(2\pi i/3).$$

Lemma 8.2.1. *For* $|ab| < 1$,

$$F_0 + F_1 + F_2 = f(a, b). \tag{8.2.4}$$

Proof. In (8.1.5), let $n = 3$. Then apply (8.1.6) and thereby deduce (8.2.4). $\square$

Theorem 8.2.1. *For* $|ab| < 1$,

$$\Big(f(a^3b^6, a^6b^3) + a\, f(b^3, a^9b^6)\Big)^3 = \frac{f(a^3, b^3)}{f(a^3, a^6b^9)} f^3(a, a^2b^3) - b^3\, f^3(a^3, a^6b^9).$$

Proof. In (8.2.4), replace b by $\omega^n b$ for $n = 0, 1, 2$, and then multiply the three identities together. We therefore find that

$$(F_0 + F_1)^3 + F_2^3 = f(a, b)f(a, \omega b)f(a, \omega^2 b) = \frac{f(a^3, b^3)}{f(a^3, a^6b^9)} f^3(a, a^2b^3),$$

by Theorem 8.1.5, with $p = 3$ and $s = 0$. Using (8.2.1)–(8.2.3), we complete the proof. $\square$

The proof of the following theorem is similar.

Theorem 8.2.2. *For* $|ab| < 1$,

$$\Big(a\, f(b^3, a^9b^6) + b\, f(a^3, a^6b^9)\Big)^3 = \frac{f(a^3, b^3)}{f(a^3b^6, a^6b^3)} f^3(ab^2, a^2b) - f^3(a^3b^6, a^6b^3).$$

Proof. In (8.2.4), replacing (a, b) by $(\omega^n a, \omega^n b)$ for $n = 0, 1, 2$, and then multiplying the three resulting identities together, we deduce that

$$\begin{aligned} F_0^3 + (F_1 + F_2)^3 &= f(a, b)f(\omega a, \omega b)f(\omega^2 a, \omega^2 b) \\ &= \frac{f(a^3, b^3)}{f(a^3b^6, a^6b^3)} f^3(ab^2, a^2b), \end{aligned}$$

by Theorem 8.1.5, with $p = 3$ and $s = 1$. Again, using the definitions (8.2.1)–(8.2.3), we complete the proof. $\square$

Theorem 8.2.3. *For* $|ab| < 1$,

$$f^3(ab^2, a^2b) - b\, f^3(a, a^2b^3) = \frac{f(-b^2, -a^3b)}{f(b, a^3b^2)} f^3(-ab).$$

Proof. In (8.1.7), replace (λ, x) by $(-a, -\omega^n b^{1/3})$ for $n = 0, 1, 2$, and then multiply the three resulting identities together to deduce that

$$f^3(ab^2, a^2b) - b\, f^3(a, a^2b^3) = \prod_{n=0}^{2} \frac{f(-\omega^{2n}b^{2/3}, -a\omega^n b^{1/3})f(-ab)}{f(-\omega^n b^{1/3}, -a\omega^{2n}b^{2/3})}. \tag{8.2.5}$$

Applying Theorem 8.1.5 twice, we find that the right side of (8.2.5) is equal to

$$\frac{f(-b^2,-a^3b)}{f(b,a^3b^2)}f^3(-ab),$$

which yields the desired result. □

We now use the results quoted or established above to prove some results on pages 54 and 48 in Ramanujan's lost notebook [244].

Entry 8.2.1 (p. 54). *Let* $A := f(-q^7,-q^8)$ *and* $B := q\,f(-q^2,-q^{13})$. *Then*

$$A+B = \frac{f(-q^2,-q^3)}{f(-q,-q^4)}f(-q^5), \tag{8.2.6}$$

$$A-B = f(-q^{2/3},-q)+q^{2/3}f(-q^3,-q^{12}), \tag{8.2.7}$$

$$A^3+B^3 = \frac{f(-q^6,-q^9)}{f(-q^3,-q^{12})}f^3(-q^5), \tag{8.2.8}$$

$$(A-B)^3 = f(-q^2,-q^3)\frac{f^3(-q,-q^4)}{f(-q^3,-q^{12})}+q^2\,f^3(-q^3,-q^{12}). \tag{8.2.9}$$

Proof. The identities (8.2.6)–(8.2.8) can be found in Ramanujan's second notebook [243]. More precisely, (8.2.6) and (8.2.7) are Entries 10(i), (iii), respectively, in Chapter 20; see [54, pp. 379–380] for statements and proofs. Also, (8.2.8) is identical to an identity on page 321 in Ramanujan's second notebook; see Entry 36(i) in [56, p. 188]. Letting $a=-q$ and $b=-q^{2/3}$ in Theorem 8.2.1 yields (8.2.9). □

After stating Entry 8.2.1, Ramanujan writes "A^3-B^3 see note." Either Ramanujan never wrote this note, or if he did, it has not been preserved.

Entry 8.2.2 (p. 54). *Let* $A := f(-q^4,-q^{11})$ *and* $B := q\,f(-q,-q^{14})$. *Then for* $|q|<1$,

$$A-B = \frac{f(-q,-q^4)}{f(-q^2,-q^3)}f(-q^5), \tag{8.2.10}$$

$$A+B = -\frac{1}{q^{1/3}}\{f(-q^{1/3},-q^{4/3})-f(-q^6,-q^9)\}, \tag{8.2.11}$$

$$A^3-B^3 = \frac{f(-q^3,-q^{12})}{f(-q^6,-q^9)}f^3(-q^5), \tag{8.2.12}$$

and

$$(A+B)^3 = -\frac{1}{q}\left\{f(-q,-q^4)\frac{f^3(-q^2,-q^3)}{f(-q^6,-q^9)}-f^3(-q^6,-q^9)\right\}. \tag{8.2.13}$$

Proof. The identities (8.2.10) and (8.2.11) can be found as Entries 10(ii), (iv) in Chapter 20 of Ramanujan's second notebook [243]; see [54, pp. 379–380]. The identity (8.2.12) is also in Ramanujan's second notebook; see Entry 36(ii) in [56, p. 188]. Putting $a = -q^{4/3}$ and $b = -q^{1/3}$ in Theorem 8.2.2, we deduce (8.2.13). □

On page 54, Ramanujan wrote "$A^3 + B^3$ see note." However, as before, he apparently did not write this note, or, if he did, it has been lost.

The next two identities are connected with what Son has called Ramanujan's theorems on circular summation found on page 54 in the lost notebook.

Entry 8.2.3 (p. 54). *For* $|ab| < 1$,

$$\begin{aligned} f^3(ab^2, a^2b) + af^3(b, a^3b^2) + bf^3(a, a^2b^3) &= f(a,b)\left\{\frac{f^9(-q)}{f^3(-q^3)} + 27q\frac{f^9(-q^3)}{f^3(-q)}\right\}^{1/3} \\ &= f(a,b)\left\{\frac{\psi^3(q)}{\psi(q^3)} + 3q\frac{\psi^3(q^3)}{\psi(q)}\right\}, \end{aligned} \tag{8.2.14}$$

where $q = ab$.

Proof. In (8.1.9), replace (a,b) by $(a^{1/3}, b^{1/3})$ to arrive at

$$\begin{aligned} f^3(ab^2, a^2b) &+ af^3(b, a^3b^2) + bf^3(a, a^2b^3) \\ &= \frac{f(a,b)q^{1/3}f^3(-q^3)}{f(-q)}\left(\frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)} + 3\right). \end{aligned} \tag{8.2.15}$$

By Entry 1(iv) in [54, p. 345],

$$\frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)} + 3 = \left\{\frac{f^{12}(-q)}{qf^{12}(-q^3)} + 27\right\}^{1/3}. \tag{8.2.16}$$

By Entry 3(i) in [54, p. 460],

$$\left\{\frac{f^9(-q)}{f^3(-q^3)} + 27q\frac{f^9(-q^3)}{f^3(-q)}\right\}^{1/3} = \frac{\psi^3(q)}{\psi(q^3)} + 3q\frac{\psi^3(q^3)}{\psi(q)}. \tag{8.2.17}$$

We now see that these three identities (8.2.15)–(8.2.17) establish (8.2.14). □

On page 48, Ramanujan presented two further cubic identities, (8.2.18)–(8.2.19) below. These identities give new representations for the Rogers–Ramanujan functions.

Entry 8.2.4 (p. 48). *For* $|q| < 1$, *let*

$$U := \sum_{n=0}^{\infty} \frac{q^{3n^2}}{(q^3;q^3)_n} \quad \text{and} \quad V := \sum_{n=0}^{\infty} \frac{q^{3n(n+1)}}{(q^3;q^3)_n}.$$

Then

$$f^3(-q^{13},-q^{17})+q^3f^3(-q^7,-q^{23})=Uf(-q^6)\frac{f^3(-q^{10})}{f(-q^{30})} \tag{8.2.18}$$

and

$$f^3(-q^{11},-q^{19})+q^9f^3(-q,-q^{29})=Vf(-q^6)\frac{f^3(-q^{10})}{f(-q^{30})}. \tag{8.2.19}$$

Proof. By the Rogers–Ramanujan identities in Theorem 8.1.6,

$$U=\frac{1}{(q^3;q^{15})_\infty(q^{12};q^{15})_\infty}. \tag{8.2.20}$$

In Theorem 8.2.3, let $a=-q^7$ and $b=-q^3$. Then

$$f^3(-q^{13},-q^{17})+q^3f^3(-q^7,-q^{23})=\frac{f(-q^6,-q^{24})}{f(-q^3,-q^{27})}f^3(-q^{10}). \tag{8.2.21}$$

Employing the Jacobi triple product identity (8.1.1), (8.1.4), and (8.2.20), we find that

$$\begin{aligned}
&\frac{f(-q^6,-q^{24})}{f(-q^3,-q^{27})}f^3(-q^{10})\\
&\qquad=\frac{(q^6;q^{30})_\infty(q^{24};q^{30})_\infty f^3(-q^{10})}{(q^3;q^{30})_\infty(q^{27};q^{30})_\infty}\\
&\qquad=\frac{(q^6;q^{30})_\infty(q^{12};q^{30})_\infty(q^{18};q^{30})_\infty(q^{24};q^{30})_\infty(q^{30};q^{30})_\infty f^3(-q^{10})}{(q^3;q^{30})_\infty(q^{12};q^{30})_\infty(q^{18};q^{30})_\infty(q^{27};q^{30})_\infty(q^{30};q^{30})_\infty}\\
&\qquad=\frac{(q^6;q^6)_\infty f^3(-q^{10})}{(q^3;q^{15})_\infty(q^{12};q^{15})_\infty f(-q^{30})}\\
&\qquad=U\frac{f(-q^6)f^3(-q^{10})}{f(-q^{30})}.
\end{aligned}$$

Using this last calculation in (8.2.21), we reach the desired result (8.2.18).

The proof of (8.2.19) is similar to that of (8.2.18). By Theorem 8.1.6,

$$V=\frac{1}{(q^6;q^{15})_\infty(q^9;q^{15})_\infty}. \tag{8.2.22}$$

In Theorem 8.2.3, let $a=-q$ and $b=-q^9$. Then

$$f^3(-q^{11},-q^{19})+q^9f^3(-q,-q^{29})=\frac{f(-q^{12},-q^{18})}{f(-q^9,-q^{21})}f^3(-q^{10}). \tag{8.2.23}$$

By the Jacobi triple product identity (8.1.1), (8.1.4), and (8.2.22),

$$\frac{f(-q^{12},-q^{18})}{f(-q^9,-q^{21})}f^3(-q^{10})$$

$$
\begin{aligned}
&= \frac{(q^{12};q^{30})_\infty(q^{18};q^{30})_\infty f^3(-q^{10})}{(q^9;q^{30})_\infty(q^{21};q^{30})_\infty} \\
&= \frac{(q^6;q^{30})_\infty(q^{12};q^{30})_\infty(q^{18};q^{30})_\infty(q^{24};q^{30})_\infty(q^{30};q^{30})_\infty f^3(-q^{10})}{(q^6;q^{30})_\infty(q^9;q^{30})_\infty(q^{21};q^{30})_\infty(q^{24};q^{30})_\infty(q^{30};q^{30})_\infty} \\
&= \frac{(q^6;q^6)_\infty f^3(-q^{10})}{(q^6;q^{15})_\infty(q^9;q^{15})_\infty f(-q^{30})} \\
&= V\frac{f(-q^6)f^3(-q^{10})}{f(-q^{30})}.
\end{aligned}
$$

Using this last identity in (8.2.23), we complete the proof of (8.2.19). □

Son [266] used results in this section to derive some new modular equations.

8.3 Septic Identities

On page 206 in his lost notebook [244], Ramanujan recorded the following identities.

Entry 8.3.1 (p. 206). *Let*

$$
\frac{\varphi(q^{1/7})}{\varphi(q^7)} = 1 + u + v + w. \tag{8.3.1}
$$

Then

$$
p := uvw = 8q^2\frac{(-q;q^2)_\infty}{(-q^7;q^{14})_\infty^7} \tag{8.3.2}
$$

and

$$
\frac{\varphi^8(q)}{\varphi^8(q^7)} - (2+5p)\frac{\varphi^4(q)}{\varphi^4(q^7)} + (1-p)^3 = 0. \tag{8.3.3}
$$

Furthermore,

$$
u = \left(\frac{\alpha^2 p}{\beta}\right)^{1/7}, \quad v = \left(\frac{\beta^2 p}{\gamma}\right)^{1/7}, \quad \textit{and} \quad w = \left(\frac{\gamma^2 p}{\alpha}\right)^{1/7}, \tag{8.3.4}
$$

where α, β, *and* γ *are roots of the equation*

$$
\xi^3 + 2\xi^2\left(1 + 3p - \frac{\varphi^4(q)}{\varphi^4(q^7)}\right) + \xi p^2(p+4) - p^4 = 0. \tag{8.3.5}
$$

For example,

$$
\varphi(e^{-7\pi\sqrt{7}}) = 7^{3/4}\varphi(e^{-\pi\sqrt{7}})\left\{1 + (-)^{2/7} + (-)^{2/7} + (-)^{2/7}\right\}. \tag{8.3.6}
$$

Although u, v, and w are not clearly defined in the claims above, their definitions can be deduced from Entry 17(iii) of Chapter 19 of Ramanujan's second notebook [54, p. 303]. In examining (8.3.1) and Entry 17(iii), we see that we are led to define

$$u := 2q^{1/7} f(q^5, q^9)/\varphi(q^7), \tag{8.3.7}$$

$$v := 2q^{4/7} f(q^3, q^{11})/\varphi(q^7), \tag{8.3.8}$$

$$w := 2q^{9/7} f(q, q^{13})/\varphi(q^7). \tag{8.3.9}$$

In (8.3.6), Ramanujan might have attempted to evaluate the quotient

$$\frac{\varphi(e^{-\pi\sqrt{7}})}{\varphi(e^{-7\pi\sqrt{7}})}, \tag{8.3.10}$$

by using the identities (8.3.1)–(8.3.5). Recall that the class invariant G_n is defined by (8.1.10). Since $G_7 = 2^{1/4}$ [57, p. 189], the value of G_{7^3} could easily be evaluated by a routine calculation if the quotient (8.3.10) were known. It is unclear why Ramanujan did not record the missing terms in (8.3.6). Did he not record them because they were inelegant? Did Ramanujan conjecture the existence of three quantities that would ensure an identity of the type (8.3.6)? We have been unsuccessful in finding the three missing terms and consequently cannot answer these questions.

In this section, our goal is to prove the identities (8.3.2)–(8.3.5). We employ modular equations of degree seven, the Jacobi triple product identity, several Lambert series identities, and the product formula for theta functions given in Theorem 8.1.5.

In the sequel, the following three Lambert series identities are needed.

Lemma 8.3.1. *For* $|q| < 1$,

(i) $$\varphi^4(q) = 1 + 8\sum_{k=1}^{\infty} \frac{kq^k}{1+(-q)^k},$$

(ii) $$\varphi^6(q) = 1 - 4\sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^2 q^{2k+1}}{1-q^{2k+1}} + 16\sum_{k=1}^{\infty} \frac{k^2 q^k}{1+q^{2k}},$$

(iii) $$\varphi^8(q) = 1 + 16\sum_{k=1}^{\infty} \frac{k^3 q^k}{1-(-q)^k}.$$

For proofs of (i)–(iii), see [54, p. 114, Entry 8(ii)], [31, p. 396, Entry 18.2.2], and [52, p. 139, Example (i)], respectively. See also [59, Chapter 3].

We recall some notation and definitions in the theory of modular equations of degree $n \geq 2$. Let β have degree n over α. Let

$$z_1 := {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right) \qquad \text{and} \qquad z_n = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \beta\right).$$

We quote from Ramanujan's catalogue of theta function evaluations in [54, pp. 122, 124].

Lemma 8.3.2. *If*

$$q = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 1-\alpha\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; \alpha\right)}\right),$$

then

$$\varphi(q) = \sqrt{z_1}, \tag{8.3.11}$$

$$\varphi(q^7) = \sqrt{z_7}, \tag{8.3.12}$$

$$\chi(q) = 2^{1/6} q^{1/24} \big(\alpha(1-\alpha)\big)^{-1/24}, \tag{8.3.13}$$

$$\chi(q^7) = 2^{1/6} q^{7/24} \big(\beta(1-\beta)\big)^{-1/24}. \tag{8.3.14}$$

For proofs of (8.3.11)–(8.3.14), see [54, pp. 122, 124].

The multiplier m of degree n is given by

$$m := \frac{\varphi^2(q)}{\varphi^2(q^n)} = \frac{z_1}{z_n}. \tag{8.3.15}$$

Lemma 8.3.3. *For* $|q| < 1$,

$$1 - 4\sum_{k=0}^{\infty} \frac{(-1)^k (2k+1)^2 q^{2k+1}}{1-q^{2k+1}} = \varphi^6(q)(1-\alpha).$$

See [54, p. 138, Entry 17(vii)], where a proof can be found.

Lemma 8.3.4. *If*

$$t := (\alpha\beta)^{1/8},$$

then

$$\big((1-\alpha)(1-\beta)\big)^{1/8} = 1 - t, \tag{8.3.16}$$

$$m - \frac{7}{m} = -(1-2t)^3 - 5(1-2t), \tag{8.3.17}$$

$$\frac{\alpha}{t} - \frac{1-\alpha}{1-t} = \frac{7}{m}\big(1 - t(1-t)\big). \tag{8.3.18}$$

For a proof of (8.3.16), see [54, p. 314, Entry 19(i)]. Using (8.3.16) along with [54, p. 315, Entry 19(viii); p. 314, Entry 19(iii)], we deduce (8.3.17) and (8.3.18), respectively.

Throughout the remainder of this section, $n = 7$, and so m is the multiplier of degree 7, β has degree 7 over α, and z_1, z_7, and t are defined as above.

Lemma 8.3.5. *For* $|q| < 1$,

$$p = 8q^2 \frac{(-q;q^2)_\infty}{(-q^7;q^{14})_\infty^7} = 8q^2 \frac{\chi(q)}{\chi^7(q^7)} = 4\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24}. \tag{8.3.19}$$

Proof. By the Jacobi triple product identity (8.1.1), the equality

$$\varphi(q^7) = f(q^7,q^7) = (-q^7;q^{14})_\infty^2 (q^{14};q^{14})_\infty,$$

and (8.3.7)–(8.3.9), we find that

$$\begin{aligned} p = uvw &= \frac{8q^2}{\varphi^3(q^7)} f(q^5,q^9) f(q^3,q^{11}) f(q,q^{13}) \frac{f(q^7,q^7)}{f(q^7,q^7)} \\ &= \frac{8q^2}{\varphi^4(q^7)} (-q;q^{14})_\infty (-q^3;q^{14})_\infty (-q^5;q^{14})_\infty (-q^7;q^{14})_\infty^2 \\ &\quad \times (-q^9;q^{14})_\infty (-q^{11};q^{14})_\infty (-q^{13};q^{14})_\infty (q^{14};q^{14})_\infty^4. \end{aligned}$$

Since

$$\begin{aligned} (-q;q^2)_\infty = (-q;q^{14})_\infty (-q^3;q^{14})_\infty (-q^5;q^{14})_\infty (-q^7;q^{14})_\infty \\ \times (-q^9;q^{14})_\infty (-q^{11};q^{14})_\infty (-q^{13};q^{14})_\infty, \end{aligned}$$

we can simplify the formula for p above to deduce that

$$p = 8q^2 \frac{(-q;q^2)_\infty}{(-q^7;q^{14})_\infty^7} = 8q^2 \frac{\chi(q)}{\chi^7(q^7)}.$$

Using (8.3.13) and (8.3.14), we complete the proof of (8.3.19). □

Note that we have also established (8.3.2).

Lemma 8.3.6. *For* $|q| < 1$,

$$1 - 2t = \frac{1-p}{m}.$$

Proof. Using Lemma 8.3.5 and (8.3.16) in Entry 19 (ii) of [54, p. 314], we complete the proof. □

We are now ready to prove (8.3.3).

Theorem 8.3.1. *Equality* (8.3.3) *is valid.*

Proof. By (8.3.17) and Lemma 8.3.6, we find that

$$m - \frac{7}{m} = -\left(\frac{1-p}{m}\right)^3 - 5\left(\frac{1-p}{m}\right).$$

Multiplying both sides by m^3, we deduce that

$$m^4 - (2+5p)m^2 + (1-p)^3 = 0.$$

Thus, by (8.3.15), we complete the proof. □

Lemma 8.3.7. *For* $|q| < 1$,

$$\varphi^4(q) = 1 + 8\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{(n-1)(k-1)}k\,q^{nk}.$$

Proof. By Lemma 8.3.1(i),

$$\begin{aligned}\varphi^4(q) &= 1 + 8\sum_{k=1}^{\infty}\frac{kq^k}{1+(-q)^k}\\ &= 1 + 8\sum_{k=1}^{\infty}kq^k\sum_{n=0}^{\infty}(-1)^n(-q)^{nk}\\ &= 1 + 8\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{n(k+1)}kq^{(n+1)k}.\end{aligned}$$

Replacing n by $n-1$, we complete the proof. □

For convenience, we define the operator $\mathcal{M}_0$, which collects the terms whose exponents are integers.

Lemma 8.3.8. *For* $|q| < 1$,

$$\mathcal{M}_0\left(\varphi^4(q^{1/7})\right) = 8\varphi^4(q) - 7\varphi^4(q^7).$$

Proof. Let

$$U_k := \sum_{n=1}^{\infty}(-1)^{(n-1)(k-1)}k\,(q^{1/7})^{nk}.$$

By Lemma 8.3.7,

$$\begin{aligned}\mathcal{M}_0\left(\varphi^4(q^{1/7})\right) &= 1 + 8\mathcal{M}_0\left(\sum_{k=1}^{\infty}U_k\right)\\ &= 1 + 8\mathcal{M}_0\left(\sum_{\substack{k=1\\ k\equiv 0\,(\mathrm{mod}\,7)}}^{\infty}U_k + \cdots + \sum_{\substack{k=1\\ k\equiv 6\,(\mathrm{mod}\,7)}}^{\infty}U_k\right)\\ &=: 1 + 8(I_0 + \cdots + I_6). \end{aligned} \tag{8.3.20}$$

In the sum with $k \equiv 0\,(\mathrm{mod}\,7)$, replace k by $7k$ to deduce that

$$\begin{aligned}I_0 &= \mathcal{M}_0\left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{(n-1)(7k-1)}(7k)(q^{1/7})^{n(7k)}\right)\\ &= 7\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{(n-1)(k-1)}kq^{nk}.\end{aligned} \tag{8.3.21}$$

For $1 \le j \le 6$,

$$\begin{aligned}
I_j &= \mathcal{M}_0 \left(\sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{(n-1)(7k+j-1)} (7k+j)(q^{1/7})^{n(7k+j)} \right) \\
&= \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{(7n-1)(7k+j-1)} (7k+j)(q^{1/7})^{(7n)(7k+j)} \\
&= \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{(n-1)(7k+j-1)} (7k+j) q^{n(7k+j)}. \qquad (8.3.22)
\end{aligned}$$

Define

$$I_0' := \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{(n-1)(7k-1)} (7k) q^{n(7k)}. \qquad (8.3.23)$$

Then, by (8.3.22) and (8.3.23),

$$I_1 + \cdots + I_6 + I_0' = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{(n-1)(k-1)} k q^{nk}. \qquad (8.3.24)$$

Employing (8.3.21), (8.3.24), and Lemma 8.3.7, we deduce that

$$I_0 + \cdots + I_6 = 8 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{(n-1)(k-1)} k q^{nk} - I_0' = \varphi^4(q) - 1 - I_0'. \qquad (8.3.25)$$

Therefore, by (8.3.20), (8.3.25), (8.3.23), and Lemma 8.3.7,

$$\begin{aligned}
\mathcal{M}_0 \left(\varphi^4(q^{1/7}) \right) &= 1 + 8(I_0 + \cdots + I_6) \\
&= 1 + 8(\varphi^4(q) - 1 - I_0') \\
&= 8\varphi^4(q) - 7 - 8I_0' \\
&= 8\varphi^4(q) - 7 - 8 \cdot 7 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{(n-1)(k-1)} k (q^7)^{nk} \\
&= 8\varphi^4(q) - 7\varphi^4(q^7),
\end{aligned}$$

which completes the proof. $\square$

Theorem 8.3.2. *Let u, v, and w be defined by* (8.3.7)–(8.3.9). *Then*

$$u^3 v + v^3 w + w^3 u = -2 \left(1 + 3p - \frac{\varphi^4(q)}{\varphi^4(q^7)} \right).$$

Proof. By (8.3.1),

$$\mathcal{M}_0 \left(\left(\frac{\varphi(q^{1/7})}{\varphi(q^7)} \right)^4 \right) = \mathcal{M}_0 \left((1 + u + v + w)^4 \right). \qquad (8.3.26)$$

By Lemma 8.3.8,

$$\mathcal{M}_0\left(\left(\frac{\varphi(q^{1/7})}{\varphi(q^7)}\right)^4\right) = 8\frac{\varphi^4(q)}{\varphi^4(q^7)} - 7. \tag{8.3.27}$$

By the multinomial theorem and (8.3.2),

$$\mathcal{M}_0\left((1+u+v+w)^4\right) = 4(u^3v+v^3w+w^3u)+24p+1. \tag{8.3.28}$$

Using (8.3.27) and (8.3.28) in (8.3.26), we deduce that

$$u^3v+v^3w+w^3u = 2\left(\frac{\varphi^4(q)}{\varphi^4(q^7)} - 3p - 1\right),$$

which is the desired result. □

Lemma 8.3.9. *For* $|q|<1$,

$$16m^3\alpha = 8m^3 + (p+20)m^2 + 7(3p+4)(p-1).$$

Proof. If we solve the linear equation (8.3.18) for α, we find that

$$\alpha = t\left(1 - \frac{7}{m}(t-1)(t^2-t+1)\right).$$

By Lemma 8.3.6,

$$t = \frac{m+p-1}{2m},$$

and so

$$\alpha = \frac{1}{16m^5}\left(8m^5 + (8p+13)m^4 - 14(p-1)^2m^2 - 7(p-1)^4\right).$$

Thus, by the equality

$$7(p-1)^4 = 7(p-1)(m^4-(2+5p)m^2),$$

which is a reformulation of (8.3.3), we find that

$$\alpha = \frac{1}{16m^3}\left(8m^3 + (p+20)m^2 + 7(3p+4)(p-1)\right),$$

which is the required result. □

Theorem 8.3.3. *For* $|q|<1$,

$$256\sum_{k=1}^{\infty}\frac{k^2q^k}{1+q^{2k}} = 8\varphi^6(q) + (p+20)\varphi^4(q)\varphi^2(q^7) + 7(3p+4)(p-1)\varphi^6(q^7).$$

Proof. By Lemma 8.3.3,

$$1-4\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)^2q^{2k+1}}{1-q^{2k+1}}=\varphi^6(q)(1-\alpha).$$

Thus, by Lemma 8.3.1(ii),

$$16\sum_{k=1}^{\infty}\frac{k^2q^k}{1+q^{2k}}=\varphi^6(q)\alpha.$$

Applying Lemma 8.3.9, we arrive at

$$\frac{256}{\varphi^6(q^7)}\sum_{k=1}^{\infty}\frac{k^2q^k}{1+q^{2k}}=16m^3\alpha=8m^3+(p+20)m^2+7(3p+4)(p-1),$$

which is equivalent to the desired result. □

Lemma 8.3.10. *For* $|q|<1$,

$$\varphi^6(q)=1-4\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}(-1)^k(2k+1)^2q^{n(2k+1)}+16\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}(-1)^kn^2q^{n(2k+1)}.$$

Proof. We shall see that the result follows readily from Lemma 8.3.1(ii). Now,

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^k(2k+1)^2q^{2k+1}}{1-q^{2k+1}}&=\sum_{k=0}^{\infty}(-1)^k(2k+1)^2q^{2k+1}\sum_{n=0}^{\infty}q^{n(2k+1)}\\&=\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}(-1)^k(2k+1)^2q^{n(2k+1)}\end{aligned}\tag{8.3.29}$$

and

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{k^2q^k}{1+q^{2k}}&=\sum_{k=1}^{\infty}k^2q^k\sum_{n=0}^{\infty}(-1)^nq^{2nk}\\&=\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(-1)^nk^2q^{k(2n+1)}\\&=\sum_{k=0}^{\infty}\sum_{n=1}^{\infty}(-1)^kn^2q^{n(2k+1)},\end{aligned}\tag{8.3.30}$$

after interchanging k and n and changing the order of summation. Substituting (8.3.29) and (8.3.30) in Lemma 8.3.1(ii), we complete the proof. □

Lemma 8.3.11. *For* $|q|<1$,

$$\mathcal{M}_0\left(\varphi^6(q^{1/7})\right)=6(p+20)\varphi^4(q)\varphi^2(q^7)+7(18p^2+6p-17)\varphi^6(q^7).$$

Proof. For $k \geq 0$, let

$$U_k := \sum_{n=1}^{\infty} (-1)^k (2k+1)^2 (q^{1/7})^{n(2k+1)}$$

and

$$V_k := \sum_{n=1}^{\infty} (-1)^k n^2 (q^{1/7})^{n(2k+1)}.$$

By Lemma 8.3.10,

$$\begin{aligned}
\mathcal{M}_0\left(\varphi^6(q^{1/7})\right) &= 1 - 4\mathcal{M}_0\left(\sum_{k=0}^{\infty} U_k\right) + 16\mathcal{M}_0\left(\sum_{k=0}^{\infty} V_k\right) \\
&= 1 - 4\mathcal{M}_0\left(\sum_{\substack{k=0 \\ k\equiv \pmod 7}}^{\infty} U_k + \cdots + \sum_{\substack{k=0 \\ k\equiv 6 \pmod 7}}^{\infty} U_k\right) \\
&\quad + 16\mathcal{M}_0\left(\sum_{\substack{k=0 \\ k\equiv 0 \pmod 7}}^{\infty} V_k + \cdots + \sum_{\substack{k=0 \\ k\equiv 6 \pmod 7}}^{\infty} V_k\right) \\
&=: 1 - 4(I_0 + \cdots + I_6) + 16(J_0 + \cdots + J_6). \qquad (8.3.31)
\end{aligned}$$

Since $2(7k+j)+1 \equiv 0 \pmod 7$ when $j = 3$,

$$\begin{aligned}
I_3 &= \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} (-1)^{7k+3}\big(2(7k+3)+1\big)^2 (q^{1/7})^{n(2(7k+3)+1)} \\
&= -7^2 \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} (-1)^k (2k+1)^2 q^{n(2k+1)}. \qquad (8.3.32)
\end{aligned}$$

For $j \neq 3$,

$$\begin{aligned}
I_j &= \mathcal{M}_0\left(\sum_{k=0}^{\infty}\sum_{n=1}^{\infty} (-1)^{7k+j}\big(2(7k+j)+1\big)^2 (q^{1/7})^{n(2(7k+j)+1)}\right) \\
&= \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} (-1)^{7k+j}\big(2(7k+j)+1\big)^2 q^{n(2(7k+j)+1)}. \qquad (8.3.33)
\end{aligned}$$

Define

$$I_3' := \sum_{k=0}^{\infty}\sum_{n=1}^{\infty} (-1)^{7k+3}\big(2(7k+3)+1\big)^2 q^{n(2(7k+3)+1)}$$

$$= -7^2 \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^k (2k+1)^2 (q^7)^{n(2k+1)}. \tag{8.3.34}$$

Thus, by (8.3.33) and (8.3.34),

$$I_0 + I_1 + I_2 + I_3' + I_4 + I_5 + I_6 = \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^k (2k+1)^2 q^{n(2k+1)}. \tag{8.3.35}$$

Using (8.3.32) and (8.3.35), we then find that

$$I_0 + \cdots + I_6 = -48 \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^k (2k+1)^2 q^{n(2k+1)} - I_3'.$$

Similarly,

$$J_0 + \cdots + J_6 = 48 \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^k n^2 q^{n(2k+1)} - J_3',$$

where

$$J_3' := -7^2 \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^k n^2 (q^7)^{n(2k+1)}. \tag{8.3.36}$$

Therefore, by (8.3.31),

$$\begin{aligned} \mathcal{M}_0 \left(\varphi^6(q^{1/7}) \right) &= 1 - 4(I_0 + \cdots + I_6) + 16(J_0 + \cdots + J_6) \\ &= 1 + 4 \cdot 48 \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^k (2k+1)^2 q^{n(2k+1)} + 4I_3' \\ &\quad + 16 \cdot 48 \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^k n^2 q^{n(2k+1)} - 16J_3'. \end{aligned}$$

By two applications of Lemma 8.3.10, (8.3.34), and (8.3.36), we deduce that

$$\mathcal{M}_0 \left(\varphi^6(q^{1/7}) \right) = 49\varphi^6(q^7) - 48\varphi^6(q) + 2 \cdot 16 \cdot 48 \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^k n^2 q^{n(2k+1)}. \tag{8.3.37}$$

Employing Theorem 8.3.3, we find that

$$\begin{aligned} 2 \cdot 16 \cdot 48 \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^k n^2 q^{n(2k+1)} &= 6 \cdot 256 \sum_{n=1}^{\infty} \frac{n^2 q^n}{1+q^{2n}} \\ &= 6 \left(8\varphi^6(q) + (p+20)\varphi^4(q)\varphi^2(q^7) + 7(3p+4)(p-1)\varphi^6(q^7) \right). \end{aligned} \tag{8.3.38}$$

Thus, substituting (8.3.38) into (8.3.37), we complete the proof. □

Lemma 8.3.12. *For* $|q| < 1$,

$$\varphi^8(q) = 1 + 16\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{(n-1)k}k^3\,q^{nk}.$$

Proof. By Lemma 8.3.1(iii),

$$\begin{aligned}
\varphi^8(q) &= 1 + 16\sum_{k=1}^{\infty}\frac{k^3q^k}{1-(-q)^k} \\
&= 1 + 16\sum_{k=1}^{\infty}k^3q^k\sum_{n=0}^{\infty}(-q)^{nk} \\
&= 1 + 16\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}(-1)^{nk}k^3q^{(n+1)k}.
\end{aligned}$$

Replacing n by $n-1$, we complete the proof. □

Lemma 8.3.13. *For* $|q| < 1$,

$$\mathcal{M}_0\left(\varphi^8(q^{1/7})\right) = 344\varphi^8(q) - 343\varphi^8(q^7).$$

Proof. For $k \geq 0$, let

$$U_k := \sum_{n=1}^{\infty}(-1)^{(n-1)k}k^3\,(q^{1/7})^{nk}.$$

By Lemma 8.3.12,

$$\begin{aligned}
\mathcal{M}_0\left(\varphi^8(q^{1/7})\right) &= 1 + 16\mathcal{M}_0\left(\sum_{k=1}^{\infty}U_k\right) \\
&= 1 + 16\mathcal{M}_0\left(\sum_{\substack{k=1 \\ k\equiv 0 \,(\mathrm{mod}\, 7)}}^{\infty}U_k + \cdots + \sum_{\substack{k=1 \\ k\equiv 6 \,(\mathrm{mod}\, 7)}}^{\infty}U_k\right) \\
&=: 1 + 16(I_0 + \cdots + I_6).
\end{aligned} \tag{8.3.39}$$

In the sum with $k \equiv 0 \pmod 7$, replace k by $7k$ to obtain

$$\begin{aligned}
I_0 &= \mathcal{M}_0\left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{(n-1)(7k)}(7k)^3(q^{1/7})^{n(7k)}\right) \\
&= 7^3\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{(n-1)k}k^3q^{nk}.
\end{aligned} \tag{8.3.40}$$

For $1 \leq j \leq 6$,

$$\begin{aligned} I_j &= \mathcal{M}_0 \left(\sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{(n-1)(7k+j)} (7k+j)^3 (q^{1/7})^{n(7k+j)} \right) \\ &= \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{(7n-1)(7k+j)} (7k+j)^3 (q^{1/7})^{(7n)(7k+j)} \\ &= \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} (-1)^{(n-1)(7k+j)} (7k+j)^3 q^{n(7k+j)}. \end{aligned} \tag{8.3.41}$$

Define

$$\begin{aligned} I_0' &:= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{(n-1)(7k)} (7k)^3 q^{n(7k)} \\ &= 7^3 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{(n-1)k} k^3 (q^7)^{nk}. \end{aligned} \tag{8.3.42}$$

Thus, by (8.3.41) and (8.3.42),

$$I_1 + \cdots + I_6 + I_0' = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{(n-1)k} k^3 q^{nk}. \tag{8.3.43}$$

Using (8.3.40) and (8.3.43), we find that

$$I_0 + \cdots + I_6 = 344 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{(n-1)k} k^3 q^{nk} - I_0'. \tag{8.3.44}$$

Hence, by (8.3.39), (8.3.44), and (8.3.42),

$$\begin{aligned} \mathcal{M}_0 \left(\varphi^8(q^{1/7}) \right) &= 1 + 16(I_0 + \cdots + I_6) \\ &= 1 + 16 \left(344 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{(n-1)k} k^3 q^{nk} - I_0' \right) \\ &= 344 \left(1 + 16 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{(n-1)k} k^3 q^{nk} \right) \\ &\quad - 343 \left(1 + 16 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{(n-1)k} k^3 (q^7)^{nk} \right) \\ &= 344 \varphi^8(q) - 343 \varphi^8(q^7), \end{aligned}$$

by Lemma 8.3.12. This completes the proof. □

For convenience, we define

$$X := u^2v^3 + v^2w^3 + w^2u^3, \tag{8.3.45}$$
$$Y := uv^5 + vw^5 + wu^5, \tag{8.3.46}$$
$$Z := u^7 + v^7 + w^7. \tag{8.3.47}$$

Lemma 8.3.14. *For* $|q| < 1$,

$$10X + Y = p(m^2 + 6p + 47).$$

Proof. By (8.3.1),

$$\mathcal{M}_0\left(\left(\frac{\varphi(q^{1/7})}{\varphi(q^7)}\right)^6\right) = \mathcal{M}_0\left((1+u+v+w)^6\right). \tag{8.3.48}$$

By Lemma 8.3.11,

$$\mathcal{M}_0\left(\left(\frac{\varphi(q^{1/7})}{\varphi(q^7)}\right)^6\right) = 6(p+20)m^2 + 7(18p^2 + 6p - 17). \tag{8.3.49}$$

By the multinomial theorem and Theorem 8.3.2,

$$\begin{aligned}\mathcal{M}_0&\left((1+u+v+w)^6\right)\\ &= 6(uv^5 + vw^5 + wu^5) + 90u^2v^2w^2 + 60(u^2v^3 + v^2w^3 + w^2u^3)\\ &\quad + 60(u^3v + v^3w + w^3u) + 120uvw + 1\\ &= 60X + 6Y + 90p^2 - 120(3p + 1 - m^2) + 120p + 1. \end{aligned}\tag{8.3.50}$$

Using (8.3.49), (8.3.50), and (8.3.2) in (8.3.48), we finish the proof. □

Lemma 8.3.15. *For* $|q| < 1$,

$$14X - 7Y + Z = m^4 - 14(p-1)m^2 + 49p^2 - 42p - 15.$$

Proof. Let $\zeta := \exp(2\pi i/7)$. Replacing $q^{1/7}$ by $\zeta q^{1/7}$ in (8.3.1), multiplying together the seven identities, and recalling the definitions of u, v, and w in (8.3.7)–(8.3.9), we find that

$$\frac{1}{\varphi^7(q^7)}\prod_{k=1}^{7}\varphi(\zeta q^{1/7}) = \prod_{k=1}^{7}\left(1 + \zeta^k u + \zeta^{4k}v + \zeta^{9k}w\right). \tag{8.3.51}$$

By Theorem 8.1.5 with $a = b = q^{1/7}$, $j = k = 1$, $s = 3$, and $p = 7$,

$$\frac{1}{\varphi^7(q^7)}\prod_{k=1}^{7}\varphi(\zeta q^{1/7}) = \frac{\varphi^8(q)}{\varphi^8(q^7)}. \tag{8.3.52}$$

Using computer algebra, we find that

$$\prod_{k=1}^{7}\left(1+\zeta^k u+\zeta^{4k}v+\zeta^{9k}w\right)$$
$$\begin{aligned}&=(u^7+v^7+w^7)+7uvw(u^3v+v^3w+w^3u)\\&\quad-7(uv^5+vw^5+wu^5)-7u^2v^2w^2\\&\quad+14(u^2v^3+v^2w^3+w^2u^3)\\&\quad-7(u^3v+v^3w+w^3u)+14uvw+1.\end{aligned}\tag{8.3.53}$$

By (8.3.51), (8.3.52), (8.3.53), (8.3.2), Theorem 8.3.2, the definitions of X, Y, and Z, and simplification, we complete the proof. $\square$

Lemma 8.3.16. *For* $|q|<1$,

$$14(2p+5)X+21Y+Z=m^4+14(p+1)m^2+28p^3+105p^2+378p-15.$$

Proof. By (8.3.1),

$$\mathcal{M}_0\left(\left(\frac{\varphi(q^{1/7})}{\varphi(q^7)}\right)^8\right)=\mathcal{M}_0\left((1+u+v+w)^8\right).\tag{8.3.54}$$

By Lemma 8.3.13 and (8.3.3) or Theorem 8.3.1,

$$\begin{aligned}\mathcal{M}_0\left(\left(\frac{\varphi(q^{1/7})}{\varphi(q^7)}\right)^8\right)&=344m^4-343=120m^4+224m^4-343\\&=120m^4+224\left((5p+2)m^2+(p-1)^3\right)-343.\end{aligned}\tag{8.3.55}$$

By the multinomial theorem,

$$\begin{aligned}&\mathcal{M}_0\left((1+u+v+w)^8\right)\\&\quad=28(u^6v^2+v^6w^2+w^6u^2)+280uvw(u^2v^3+v^2w^3+w^2u^3)\\&\qquad+8(u^7+v^7+w^7)+840uvw(u^3v+v^3w+w^3u)\\&\qquad+168(uv^5+vw^5+wu^5)+2520u^2v^2w^2\\&\qquad+560(u^2v^3+v^2w^3+w^2u^3)\\&\qquad+280(u^3v+v^3w+w^3u)+336uvw+1.\end{aligned}\tag{8.3.56}$$

By Theorem 8.3.2,

$$(u^3v+v^3w+w^3u)^2=4(3p+1-m^2)^2.\tag{8.3.57}$$

Thus, by (8.3.57) and (8.3.2),

$$u^6v^2+v^6w^2+w^6u^2=4(3p+1-m^2)^2-2pX.\tag{8.3.58}$$

By (8.3.54), (8.3.55), (8.3.56), (8.3.58), (8.3.2), Theorem 8.3.2, and simplification, we complete the proof. $\square$

Theorem 8.3.4. *Let u, v, and w be defined by* (8.3.7)–(8.3.9). *Then*

$$u^2v^3 + v^2w^3 + w^2u^3 = p(p+4), \tag{8.3.59}$$

$$uv^5 + vw^5 + wu^5 = p\left(\frac{\varphi^4(q)}{\varphi^4(q^7)} - 4p + 7\right),$$

$$u^7 + v^7 + w^7 = \frac{\varphi^8(q)}{\varphi^8(q^7)} - 7(p-2)\frac{\varphi^4(q)}{\varphi^4(q^7)} + 7p^2 - 49p - 15.$$

Proof. Lemmas 8.3.14–8.3.16 give a system of linear equations in the variables X, Y, and Z. Solving this linear system for X, Y, and Z and recalling the definitions of X, Y, and Z in (8.3.45)–(8.3.47), we complete the proof. □

We are now in a position to prove (8.3.5).

Theorem 8.3.5. *Equality* (8.3.5) *is valid.*

Proof. Without loss of generality, we can assume that

$$\alpha := u^3v, \qquad \beta := v^3w, \qquad \text{and} \qquad \gamma := w^3u. \tag{8.3.60}$$

Solving the system (8.3.60) for u, v, and w, we obtain (8.3.4). Using Theorem 8.3.2, (8.3.59), and (8.3.2), we find that

$$\alpha + \beta + \gamma = -2(1 + 3p - m^2),$$
$$\alpha\beta + \beta\gamma + \gamma\alpha = p^2(p+4),$$
$$\alpha\beta\gamma = p^4.$$

Thus α, β, and γ are roots of the equation (8.3.5). □

9

Ramanujan's Cubic Analogue of the Classical Ramanujan–Weber Class Invariants

9.1 Introduction

At the top of page 212 in his lost notebook [244], Ramanujan defines the function λ_n by

$$\lambda_n = \frac{e^{\pi/2\sqrt{n/3}}}{3\sqrt{3}}\{(1+e^{-\pi\sqrt{n/3}})(1-e^{-2\pi\sqrt{n/3}})(1-e^{-4\pi\sqrt{n/3}})\cdots\}^6, \tag{9.1.1}$$

and then devotes the remainder of the page to stating several elegant values of λ_n, for $n \equiv 1 \pmod 8$. The quantity λ_n can be thought of as an analogue in Ramanujan's cubic theory of elliptic functions [57, Chapter 33] of the classical Ramanujan–Weber class invariant G_n, which is defined by

$$G_n := 2^{-1/4}q^{-1/24}(-q;q^2)_\infty, \tag{9.1.2}$$

where $q = \exp(-\pi\sqrt{n})$ and n is any positive rational number.

Entry 9.1.1 (p. 212).

$$\begin{gathered}
\lambda_1 = 1, \qquad \lambda_9 = 3, \qquad \lambda_{17} = 4+\sqrt{17}, \qquad \lambda_{25} = (2+\sqrt{5})^2, \\
\lambda_{33} = 18+3\sqrt{33}, \quad \lambda_{41} = 32+5\sqrt{41}, \qquad \lambda_{49} = 55+12\sqrt{21}, \\
\lambda_{57} =, \qquad \lambda_{65} =, \qquad \lambda_{81} =, \qquad \lambda_{89} = 500+53\sqrt{89},
\end{gathered}$$

$$\lambda_{73} = \left(\sqrt{\frac{11+\sqrt{73}}{8}} + \sqrt{\frac{3+\sqrt{73}}{8}}\right)^6,$$

$$\lambda_{97} = \left(\sqrt{\frac{17+\sqrt{97}}{8}} + \sqrt{\frac{9+\sqrt{97}}{8}}\right)^6,$$

$$\lambda_{121} = \left(\frac{3\sqrt{3}+\sqrt{11}}{4} + \sqrt{\frac{11+3\sqrt{33}}{8}}\right)^6,$$

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$$\lambda_{169} =, \qquad \lambda_{193} =, \qquad \lambda_{217} =, \qquad \lambda_{241} =,$$
$$\lambda_{265} =, \qquad \lambda_{289} =, \qquad \lambda_{361} = .$$

Note that for several values of n, Ramanujan did not record the corresponding values of λ_n.

The purpose of this chapter is to establish all the values of λ_n in Entry 9.1.1, including the ones that are not explicitly stated by Ramanujan, using the modular j-invariant, modular equations, Kronecker's limit formula, and the explicit Shimura reciprocity law.

The function λ_n had been briefly introduced in his third notebook [243, p. 393], where Ramanujan offers a formula for λ_n in terms of Klein's J-invariant, which was first proved by Berndt and H.H. Chan [63], [57, p. 318, Entry 11.21] using Ramanujan's cubic theory of elliptic functions. As K.G. Ramanathan [237] pointed out, the formula in the third notebook is for evaluating $\lambda_{n/3}$, especially for $n = 11, 19, 43, 67, 163$. Observe that $-11, -19, -43, -67$, and -163 are precisely the discriminants congruent to 5 modulo 8 of imaginary quadratic fields of class number one. (Ramanathan inadvertently inverted the roles of n and $n/3$ in his corresponding remark.) In Section 9.2, we discuss some of these results in Ramanujan's third notebook and show how they can be used to calculate the values of λ_n when $3 \mid n$.

In this and the next two paragraphs, we offer some necessary definitions. Let $\eta(\tau)$ denote the Dedekind eta function, defined by

$$\eta(\tau) := e^{2\pi i\tau/24} \prod_{n=1}^{\infty} (1 - e^{2\pi i n\tau}) =: q^{1/24} f(-q), \tag{9.1.3}$$

where $q = e^{2\pi i\tau}$ and $\operatorname{Im} \tau > 0$. Then (9.1.1) can be written in the alternative forms

$$\lambda_n = \frac{1}{3\sqrt{3}} \frac{f^6(q)}{\sqrt{q} f^6(q^3)} = \frac{1}{3\sqrt{3}} \left(\frac{\eta\left(\dfrac{1 + i\sqrt{n/3}}{2}\right)}{\eta\left(\dfrac{1 + i\sqrt{3n}}{2}\right)} \right)^6, \tag{9.1.4}$$

where $q = e^{-\pi\sqrt{n/3}}$.

Since much of this chapter is devoted to the evaluation of λ_n using modular equations, we now give a definition of a modular equation. Let $(a)_k = (a)(a+1)\cdots(a+k-1)$ and define the ordinary hypergeometric function ${}_2F_1(a, b; c; z)$ by

$${}_2F_1(a, b; c; z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}, \qquad |z| < 1.$$

Suppose that, for $0 < \alpha, \beta < 1$,

$$\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\beta\right)} = n\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\alpha\right)}, \tag{9.1.5}$$

for some positive integer n. A relation between α and β induced by (9.1.5) is called a modular equation of degree n, and β is said to have degree n over α.

In Section 9.3, using a modular equation of degree 3, we derive a formula for λ_n in terms of the Ramanujan–Weber class invariant.

In the theory of signature 3, or in Ramanujan's cubic theory, we say that β has degree n over α when

$$\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\beta\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right)} = n\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\alpha\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha\right)}. \tag{9.1.6}$$

A modular equation of degree n in the cubic theory is a relation between α and β that is induced by (9.1.6).

In Sections 9.4 and 9.5, we establish all eight values of λ_{p^2} in Entry 9.1.1, where p is a prime. Our proofs in Section 9.4 employ certain modular equations of degrees 3, 5, 7, and 11 in the theory of signature 3. The first three were claimed by Ramanujan in [244] and proved by Berndt, S. Bhargava, and F.G. Garvan in [60], and the last one was discovered more recently by the aforementioned authors and proved in [60]. In Section 9.5, we employ recent discoveries by Chan and W.-C. Liaw [108], [204] on Russell-type modular equations of degrees 13, 17, and 19 in the theory of signature 3. In these two sections, we also determine the value of λ_{17}.

In analogy to the singular modulus α_n in the classical theory of elliptic functions [57, pp. 277–309], the *cubic singular modulus* is defined to be the unique positive number α_n^* between 0 and 1 such that

$$\frac{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;1-\alpha_n^*\right)}{{}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha_n^*\right)} = \sqrt{n}, \quad n \in \mathbb{Q}.$$

Although Ramanujan did not record any cubic singular moduli in his notebooks [243] or lost notebook [244], it would appear that he used values of the cubic singular modulus to derive some of his series representations for $1/\pi$ [239] arising from his alternative cubic theory. As we shall see in Sections 9.3 and 9.8, the cubic singular modulus is intimately related to a companion μ_n of λ_n first defined by Ramanathan [237].

In two papers [236], [237], using Kronecker's limit formula, Ramanathan determined several values of λ_n. In [237], in order to determine two specific values of the Rogers–Ramanujan continued fraction, he evaluated λ_{25} by applying

Kronecker's limit formula to L-functions of orders of $\mathbb{Q}(\sqrt{-3})$ with conductor 5. This method was also used to determine λ_{49}. In the other paper [236], Ramanathan found a representation for λ_n in terms of fundamental units, where $-3n$ is a fundamental discriminant of an imaginary quadratic field $\mathbb{Q}(\sqrt{-3n})$ that has only one class in each genus of ideal classes. In particular, he calculated λ_{17}, λ_{41}, λ_{65}, λ_{89}, and λ_{265}. This formula and all 14 values of such λ_n's are given in Section 9.6. In the same section, we extend Ramanathan's method to establish a similar result for λ_n when $-3n \equiv 3 \pmod 4$ and there is precisely one class per genus in each imaginary quadratic field $\mathbb{Q}(\sqrt{-3n})$.

Through Section 9.6, all values of λ_n in Entry 9.1.1 are calculated except for $n = 73, 97, 193, 217, 241$. In [65], we employed an empirical process, analogous to that employed by G.N. Watson [275], [276] in his calculation of class invariants, to determine λ_n for these remaining values of n. This empirical method has been put on a firm foundation by Chan, A. Gee, and V. Tan [107]. Their method works whenever $3 \nmid n$, n is square-free, and the class group of $\mathbb{Q}(\sqrt{-3n})$ takes the form $\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \cdots \oplus \mathbb{Z}_{2k}$, with $1 \le k \le 4$.

The first representation of λ_n in (9.1.4) suggests connections between λ_n and Ramanujan's alternative cubic theory. In fact, Berndt and Chan [63] first found such a relationship. For other connections with the cubic theory and for recent applications of values of λ_n, see papers by Chan, Liaw, and Tan [109], Chan, Gee, and Tan [107], and Berndt and Chan [64].

Values of λ_n play an important role in generating rapidly convergent series for $1/\pi$. For example, using the value of λ_{1105}, Berndt and Chan [64] established a series for $1/\pi$ that yields about 73 or 74 digits of π per term. The previous record, which yields 50 digits of π per term, was given by the Borweins [88] in 1988. Chan, Liaw, and Tan [109] generated a new class of series for $1/\pi$ depending on values of λ_n. For example, they proved that

$$\frac{4}{\pi\sqrt{3}} = \sum_{k=0}^{\infty}(5k+1)\frac{\left(\frac{1}{3}\right)_k\left(\frac{2}{3}\right)_k\left(\frac{1}{2}\right)_k}{(1)_k^3}\left(-\frac{9}{16}\right)^k,$$

which follows from the value $\lambda_9 = 3$ and a certain Lambert-type series identity.

In Entry 9.1.1, we observe that if n is not divisible by 3, then λ_n is a unit. In fact, it can be shown that λ_n is a unit when n is odd and $3 \nmid n$ [65].

This chapter is based primarily on two papers, the first by Berndt, Chan, S.-Y. Kang, and L.-C. Zhang [65], and the second by Chan, Gee, and Tan [107]. We emphasize that in this chapter we concentrate on the values of λ_n given in Entry 9.1.1. Many further values of λ_n are determined in [65]. As remarked above, Ramanathan [237] studied a companion to λ_n, and a more extensive generalization of λ_n has been examined by M.S. Mahadeva Naika, M.C. Maheshkumar, and K. Sushan Bairy [217].

9.2 λ_n and the Modular j-Invariant

Recall [120, p. 81] that the invariants $J(\tau)$ and $j(\tau)$, for $\tau \in \mathcal{H} := \{\tau : \operatorname{Im} \tau > 0\}$, are defined by

$$J(\tau) = \frac{g_2^3(\tau)}{\Delta(\tau)} \qquad \text{and} \qquad j(\tau) = 1728J(\tau), \tag{9.2.1}$$

where

$$\Delta(\tau) = g_2^3(\tau) - 27g_3^2(\tau),$$
$$g_2(\tau) = 60 \sum_{m,n=-\infty (m,n)\neq(0,0)}^{\infty} (m\tau + n)^{-4}, \tag{9.2.2}$$

and

$$g_3(\tau) = 140 \sum_{\substack{m,n=-\infty \\ (m,n)\neq(0,0)}}^{\infty} (m\tau + n)^{-6}. \tag{9.2.3}$$

Furthermore, the function $\gamma_2(\tau)$ is defined by [146, p. 249]

$$\gamma_2(\tau) = \sqrt[3]{j(\tau)},$$

where the principal branch is chosen.

In his third notebook, at the top of page 392 in the pagination of [243], Ramanujan defines a function J_n by

$$J_n = -\frac{1}{32}\gamma_2\left(\frac{3+\sqrt{-n}}{2}\right) = -\frac{1}{32}\sqrt[3]{j\left(\frac{3+\sqrt{-n}}{2}\right)}. \tag{9.2.4}$$

For fifteen values of n, $n \equiv 3 \pmod 4$, Ramanujan indicates the corresponding fifteen values for J_n. See [57, pp. 310–312] for proofs of these evaluations. In particular,

$$J_3 = 0, \qquad J_{27} = 5 \cdot 3^{1/3}, \qquad J_{51} = 3(\sqrt{17}+4)^{2/3}\frac{5+\sqrt{17}}{2},$$
$$J_{75} = 3 \cdot 5^{1/6}\frac{69+31\sqrt{5}}{2}, \qquad J_{99} = (23+4\sqrt{33})^{2/3}\frac{77+15\sqrt{33}}{2}. \tag{9.2.5}$$

The first five values of n for λ_n in Entry 9.1.1 are those for which $3n = 3, 27, 51, 75, 99$; the corresponding five values of J_n are given in (9.2.5). Then on the next page, which is the last page of his third notebook, Ramanujan gives a formula leading to a representation of λ_n.

Theorem 9.2.1. *For $q = \exp(-\pi\sqrt{n})$, define*

$$R := R_n := 3^{1/4} q^{1/36} \frac{f(q)}{f(q^{1/3})}. \tag{9.2.6}$$

Then

$$\frac{3\sqrt{3}}{R_n^6} = \sqrt{8J_n + 3} + \sqrt{2\sqrt{64J_n^2 - 24J_n + 9} - 8J_n + 6}. \tag{9.2.7}$$

Theorem 9.2.1 was first proved in [57, p. 318, Entry 11.21]. Since $\lambda_n = R_{3n}^{-6}$ by (9.1.4) and (9.2.6), (9.2.7) may be restated in the form

$$3\sqrt{3}\lambda_{n/3} = \sqrt{8J_n + 3} + \sqrt{2\sqrt{64J_n^2 - 24J_n + 9} - 8J_n + 6}. \tag{9.2.8}$$

By substituting $J_3 = 0$ into (9.2.8), we determine the first value of λ_n in Entry 9.1.1, and we state it as a corollary.

Corollary 9.2.1 (p. 212).

$$\lambda_1 = 1.$$

Unfortunately, it is not so easy to find other values of λ_n from Theorem 9.2.1. We have to struggle with complicated radicals even when $n = 9$, for which $\lambda_9 = 3$. It seems that Ramanujan used this formula to determine the values of $\lambda_{n/3}$ for rational integral values of J_n as given in Table 9.1, which constitutes the first part of the last page of the third notebook.

n	J_n	$8J_n + 3$	$64J_n^2 - 24J_n + 9$
11	1	11	$49 = 7^2$
19	3	27	$513 = 27 \cdot 19$
43	30	$243 = 27 \cdot 3^2$	$56,889 = 27 \cdot 43 \cdot 7^2$
67	165	$1323 = 27 \cdot 7^2$	$1,738,449 = 27 \cdot 31^2 \cdot 67$
163	20010	$160,083 = 27 \cdot 77^2$	$25,625,126,169 = 27 \cdot 163 \cdot 2413^2$

Table 9.1

We use Ramanujan's discoveries recorded between Table 9.1 and Theorem 9.2.1 on the last page in his third notebook. Ramanujan first sets, for $q = \exp(-\pi\sqrt{n})$,

$$t_n := \sqrt{3} q^{1/18} \frac{f(q^{1/3}) f(q^3)}{f^2(q)} \tag{9.2.9}$$

and

$$u_n := \frac{1}{3}\sqrt{1 + \frac{8}{3}J_n}. \tag{9.2.10}$$

(To avoid a conflict of notation, we have replaced Ramanujan's second use of t_n by u_n.) He then asserts that

$$t_n = \left(2\sqrt{64J_n^2 - 24J_n + 9} - (16J_n - 3)\right)^{1/6} \tag{9.2.11}$$

and lists very simple polynomials satisfied by t_n and u_n. The definition of u_n in (9.2.10) seems unmotivated, but by recalling from the proof of Theorem 9.2.1 in [57, p. 321, Equation (11.33)] that

$$2\sqrt{8J_n + 3} = \frac{f^6(q^{1/3})}{q^{1/6}f^6(q)} - 27\sqrt{q}\frac{f^6(q^3)}{f^6(q)}, \tag{9.2.12}$$

we find that

$$2u_n = \frac{1}{3\sqrt{3}}\frac{f^6(q^{1/3})}{q^{1/6}f^6(q)} - 3\sqrt{3}\sqrt{q}\frac{f^6(q^3)}{f^6(q)}. \tag{9.2.13}$$

We summarize these results in the following two corollaries.

Corollary 9.2.2.

$$\lambda_{n/3} - \lambda_{3n}^{-1} = \frac{2\sqrt{8J_n + 3}}{3\sqrt{3}}.$$

Proof. This is a restatement of either (9.2.12) or (9.2.13), with the definition of λ_n being given by (9.1.4). □

Corollary 9.2.3.

$$\frac{\lambda_{3n}}{\lambda_{n/3}} = 2\sqrt{64J_n^2 - 24J_n + 9} + (16J_n - 3).$$

Proof. By (9.1.4) and (9.2.9), $t_n^6 = 27\lambda_{n/3}\lambda_{3n}^{-1}$. We therefore obtain the result at once from (9.2.11). □

Corollary 9.2.4 (p. 212).

$$\lambda_9 = 3.$$

Proof. Let $n = 3$ in either Corollary 9.2.2 or Corollary 9.2.3. The result follows immediately from the facts that $J_3 = 0$ from (9.2.5) and $\lambda_1 = 1$. □

Corollary 9.2.5 (p. 212).

$$\lambda_{33} = (3\sqrt{3})(2\sqrt{3} + \sqrt{11}) = 18 + 3\sqrt{33}.$$

Proof. Using Theorem 9.2.1 with $n = 11$ and the value $J_{11} = 1$ from Table 9.1, we find that

$$\lambda_{11/3} = \frac{2\sqrt{3} + \sqrt{11}}{3\sqrt{3}}. \tag{9.2.14}$$

Using (9.2.14) and either Corollary 9.2.2 or Corollary 9.2.3 when $n = 11$, we obtain Corollary 9.2.5. □

Corollary 9.2.6 (p. 212).

$$\lambda_{57} = 3^{3/4}\sqrt{2\sqrt{19}+5\sqrt{3}}\left(\sqrt{46+6\sqrt{57}}+\sqrt{45+6\sqrt{57}}\right).$$

Proof. Using (9.2.8) with $n=19$, we find that, since $J_{19}=3$ from Table 9.1,

$$\lambda_{19/3} = 3^{-3/4}\left(\sqrt{3\sqrt{3}}+\sqrt{2(\sqrt{19}-\sqrt{3})}\right).$$

Let us represent $x=\sqrt{3\sqrt{3}}+\sqrt{2(\sqrt{19}-\sqrt{3})}$ as a product of units. If $t=\sqrt{2(\sqrt{19}-\sqrt{3})}$, then $(x-t)^2=3\sqrt{3}$, or

$$x^2-2tx+2\sqrt{19}-5\sqrt{3}=0. \tag{9.2.15}$$

Let

$$y=\frac{x}{\sqrt{2\sqrt{19}-5\sqrt{3}}}.$$

Then (9.2.15) becomes

$$x\sqrt{2\sqrt{19}-5\sqrt{3}}\left(y-\frac{2t}{\sqrt{2\sqrt{19}-5\sqrt{3}}}+\frac{1}{y}\right)=0. \tag{9.2.16}$$

Hence, applying the quadratic formula to

$$y+\frac{1}{y}=\frac{2\sqrt{2\sqrt{19}-2\sqrt{3}}}{\sqrt{2\sqrt{19}-5\sqrt{3}}}=2\sqrt{46+6\sqrt{57}},$$

we find that

$$y=\sqrt{46+6\sqrt{57}}+\sqrt{45+6\sqrt{57}},$$

from which it follows that

$$\lambda_{19/3}=3^{-3/4}\sqrt{2\sqrt{19}-5\sqrt{3}}\left(\sqrt{46+6\sqrt{57}}+\sqrt{45+6\sqrt{57}}\right). \tag{9.2.17}$$

From (9.2.17) and either Corollary 9.2.2 or Corollary 9.2.3 with $n=19$, we deduce Corollary 9.2.6. □

By similar methods, we can calculate the values of λ_{129}, λ_{201}, and λ_{489} [65].

9.3 λ_n and the Class Invariant G_n

In [237], Ramanathan introduced a new function μ_n defined by

$$\mu_n := \frac{1}{3\sqrt{3}}\left(\frac{\eta(i\sqrt{n/3})}{\eta(i\sqrt{3n})}\right)^6 = \frac{1}{3\sqrt{3}}\frac{f^6(-q^2)}{qf^6(-q^6)}, \qquad q = e^{-\pi\sqrt{n/3}}. \tag{9.3.1}$$

Then by (9.1.4), (9.3.1), and Euler's pentagonal number theorem (8.1.4),

$$\frac{\lambda_n}{\mu_n} = q^{1/2}\left(\frac{f(q)f(-q^6)}{f(-q^2)f(q^3)}\right)^6 = q^{1/2}\left(\frac{(-q;q^2)_\infty}{(-q^3;q^6)_\infty}\right)^6.$$

Hence from (9.1.2), we deduce the following result.

Theorem 9.3.1.

$$\frac{\lambda_n}{\mu_n} = \left(\frac{G_{n/3}}{G_{3n}}\right)^6.$$

Let

$$P = \frac{f(-q)}{q^{1/12}f(-q^3)} \qquad \text{and} \qquad Q_p = \frac{f(-q^p)}{q^{p/12}f(-q^{3p})}. \tag{9.3.2}$$

Recall the modular equation [56, p. 204, Entry 51]

$$(PQ_2)^2 + \frac{9}{(PQ_2)^2} = \left(\frac{Q_2}{P}\right)^6 + \left(\frac{P}{Q_2}\right)^6. \tag{9.3.3}$$

Replacing q by $-q$ in (9.3.3), we deduce from (9.1.4), (9.3.1), (9.3.2), and Theorem 9.3.1 that

$$3(\lambda_n\mu_n)^{1/3} - 3(\lambda_n\mu_n)^{-1/3} = \left(\frac{G_{3n}}{G_{n/3}}\right)^6 - \left(\frac{G_{n/3}}{G_{3n}}\right)^6. \tag{9.3.4}$$

Solving (9.3.4) for $\lambda_n\mu_n$, we find that

$$\lambda_n\mu_n = \left(\frac{c+\sqrt{c^2+9}}{3}\right)^3, \tag{9.3.5}$$

where

$$2c = \left(\frac{G_{3n}}{G_{n/3}}\right)^6 - \left(\frac{G_{n/3}}{G_{3n}}\right)^6. \tag{9.3.6}$$

Hence from Theorem 9.3.1 and (9.3.5), we derive the following theorem.

Theorem 9.3.2. *If c is defined by* (9.3.6), *then*

$$\lambda_n = \left(\left(\frac{G_{n/3}}{G_{3n}}\right)\sqrt{\frac{c+\sqrt{c^2+9}}{3}}\right)^3.$$

We give another proof of Corollary 9.2.1.

Corollary 9.3.1 (p. 212).

$$\lambda_1 = 1 \qquad \textit{and} \qquad \mu_1 = 1.$$

Proof. Since $G_{1/n} = G_n$ [239], $(G_{1/3}/G_3)^6 = 1$. Substituting this value into Theorem 9.3.2, we find the value of λ_1, and then using Theorem 9.3.1, we deduce the value of μ_1 at once. □

Setting $n = 3$ in Theorem 9.3.2, we can deduce that [65]

$$\lambda_3 = 3^{3/4}\frac{\sqrt{3}-1}{\sqrt{2}}.$$

9.4 λ_n and Modular Equations

We employ a certain type of modular equation of degree p in P and Q_p (defined in (9.3.2)) to calculate several values of λ_n. First, recall the modular equation of degree 9 [54, p. 346, Entry 1(iv)]

$$1 + 9q\frac{f^3(-q^9)}{f^3(-q)} = \left(1 + 27q\frac{f^{12}(-q^3)}{f^{12}(-q)}\right)^{1/3}. \tag{9.4.1}$$

After replacing q by $-q$ on both sides, we deduce the following result from the definition of λ_n in (9.1.4).

Theorem 9.4.1.

$$1 - \frac{1}{\lambda_n^2} = \left(1 - \sqrt{\frac{3}{\lambda_n\lambda_{9n}}}\right)^3.$$

Corollary 9.4.1 (p. 212). *We have*

$$\lambda_9 = 3$$

and

$$\lambda_{81} = 3\sqrt[3]{3}(52 + 36\sqrt[3]{3} + 25\sqrt[3]{3^2}).$$

Proof. Letting $n = 1$ and $n = 9$ in Theorem 9.4.1, we obtain, respectively, the values of λ_9 and λ_{81}. □

At the end of Section 9.1, we remarked that λ_n is a unit for odd n not divisible by 3. With this as motivation, set

$$\lambda_n = (\sqrt{a+1} + \sqrt{a})^6, \tag{9.4.2}$$

and let

$$\Lambda = \lambda_n^{1/3} + \lambda_n^{-1/3}. \tag{9.4.3}$$

Then

$$a = \frac{\Lambda - 2}{4}. \tag{9.4.4}$$

By determining Λ in (9.4.3) and then using (9.4.4) and (9.4.2), we next use modular equations to prove all the evaluations of λ_{p^2} given explicitly by Ramanujan in Entry 9.1.1.

Theorem 9.4.2.

$$(27\lambda_n\lambda_{25n})^{1/3} + \left(\frac{27}{\lambda_n\lambda_{25n}}\right)^{1/3} = \left(\frac{\lambda_{25n}}{\lambda_n}\right)^{1/2} - \left(\frac{\lambda_n}{\lambda_{25n}}\right)^{1/2} + 5.$$

Proof. From [56, p. 221, Entry 62], we find that

$$(PQ_5)^2 + 5 + \frac{9}{(PQ_5)^2} = \left(\frac{Q_5}{P}\right)^3 - \left(\frac{P}{Q_5}\right)^3, \tag{9.4.5}$$

where P and Q_5 are defined by (9.3.2). We can immediately deduce Theorem 9.4.2 from (9.4.5) and (9.1.4) after replacing q by $-q$. □

Corollary 9.4.2 (p. 212).

$$\lambda_{25} = \left(\frac{1+\sqrt{5}}{2}\right)^6 = (2+\sqrt{5})^2.$$

Proof. For brevity, we set $\lambda = \lambda_{25}$ in the proof. Let $n = 1$ in Theorem 9.4.2. Then we see that

$$3(\lambda^{1/3} + \lambda^{-1/3}) = (\lambda^{1/2} - \lambda^{-1/2}) + 5. \tag{9.4.6}$$

Set $\Lambda = \lambda^{1/3} + \lambda^{-1/3}$. Since

$$\lambda^{1/2} - \lambda^{-1/2} = (\lambda^{1/6} - \lambda^{-1/6})(\lambda^{1/3} + \lambda^{-1/3} + 1),$$

(9.4.6) becomes

$$3\Lambda - 5 = (\Lambda - 2)^{1/2}(\Lambda + 1),$$

which can be simplified, after squaring both sides, to

$$(\Lambda - 3)^3 = 0.$$

Thus $\Lambda = 3$ and $a = 1/4$ by (9.4.4). Hence, from (9.4.2),

$$\lambda_{25} = \left(\sqrt{\frac{5}{4}} + \sqrt{\frac{1}{4}}\right)^6.$$

□

Theorem 9.4.3.

$$\begin{aligned}\left(27\lambda_n\lambda_{49n}\right)^{1/2}&+\left(\frac{27}{\lambda_n\lambda_{49n}}\right)^{1/2}\\&=\left(\frac{\lambda_{49n}}{\lambda_n}\right)^{2/3}+7\left(\frac{\lambda_{49n}}{\lambda_n}\right)^{1/3}-7\left(\frac{\lambda_n}{\lambda_{49n}}\right)^{1/3}-\left(\frac{\lambda_n}{\lambda_{49n}}\right)^{2/3}.\end{aligned}$$

Proof. With the use of (9.3.2) and (9.1.4), our theorem can be deduced from the modular equation [56, p. 236, Entry 69]

$$(PQ_7)^3+\frac{27}{(PQ_7)^3}=\left(\frac{Q_7}{P}\right)^4-7\left(\frac{Q_7}{P}\right)^2+7\left(\frac{P}{Q_7}\right)^2-\left(\frac{P}{Q_7}\right)^4, \tag{9.4.7}$$

with q replaced by $-q$. □

Corollary 9.4.3 (p. 212).

$$\lambda_{49}=\left(\frac{\sqrt{7}+\sqrt{3}}{2}\right)^6=55+12\sqrt{21}.$$

Proof. Let $n=1$ in Theorem 9.4.3 and set $\lambda=\lambda_{49}$ to deduce that

$$\begin{aligned}3\sqrt{3}(\lambda^{1/2}+\lambda^{-1/2})&=\lambda^{2/3}+7\lambda^{1/3}-7\lambda^{-1/3}-\lambda^{-2/3}\\&=(\lambda^{1/3}-\lambda^{-1/3})(\lambda^{1/3}+\lambda^{-1/3}+7),\end{aligned} \tag{9.4.8}$$

which can be simplified to

$$3\sqrt{3}(\lambda^{1/3}+\lambda^{-1/3}-1)=(\lambda^{1/6}-\lambda^{-1/6})(\lambda^{1/3}+\lambda^{-1/3}+7). \tag{9.4.9}$$

Letting $\Lambda=\lambda^{1/3}+\lambda^{-1/3}$ in (9.4.9), we find that

$$3\sqrt{3}(\Lambda-1)=\sqrt{\Lambda-2}(\Lambda+7). \tag{9.4.10}$$

Squaring both sides of (9.4.10), we deduce that

$$(\Lambda-5)^3(\Lambda+2)=0.$$

Hence, $\Lambda=5$ and

$$\lambda_{49}=\left(\sqrt{\frac{7}{4}}+\sqrt{\frac{3}{4}}\right)^6,$$

by (9.4.2) and (9.4.4). □

Theorem 9.4.4.

$$\begin{aligned}&9\sqrt{3}\{(\lambda_n\lambda_{121n})^{5/6}+(\lambda_n\lambda_{121n})^{-5/6}\}-99\{(\lambda_n\lambda_{121n})^{2/3}+(\lambda_n\lambda_{121n})^{-2/3}\}\\&\quad+198\sqrt{3}\{(\lambda_n\lambda_{121n})^{1/2}+(\lambda_n\lambda_{121n})^{-1/2}\}\\&\quad-759\{(\lambda_n\lambda_{121n})^{1/3}+(\lambda_n\lambda_{121n})^{-1/3}\}\\&\quad+693\sqrt{3}\{(\lambda_n\lambda_{121n})^{1/6}+(\lambda_n\lambda_{121n})^{-1/6}\}-1386=\left(\frac{\lambda_n}{\lambda_{121n}}\right)+\left(\frac{\lambda_{121n}}{\lambda_n}\right).\end{aligned}$$

Proof. A modular equation of degree 11, which was not mentioned by Ramanujan, was established by Berndt, Bhargava, and Garvan [60], and is given by

$$\begin{aligned}(PQ_{11})^5 &+ \left(\frac{3}{PQ_{11}}\right)^5 + 11\left\{(PQ_{11})^4 + \left(\frac{3}{PQ_{11}}\right)^4\right\} \\ &+ 66\left\{(PQ_{11})^3 + \left(\frac{3}{PQ_{11}}\right)^3\right\} + 253\left\{(PQ_{11})^2 + \left(\frac{3}{PQ_{11}}\right)^2\right\} \\ &+ 693\left\{(PQ_{11}) + \left(\frac{3}{PQ_{11}}\right)\right\} + 1368 = \left(\frac{P}{Q_{11}}\right)^6 + \left(\frac{Q_{11}}{P}\right)^6,\end{aligned} \tag{9.4.11}$$

where P and Q_{11} are defined by (9.3.2). Replacing q by $-q$ in the equation above, then setting $q = e^{-\pi\sqrt{n/3}}$, and lastly using (9.1.4), we deduce Theorem 9.4.4. □

Corollary 9.4.4 (p. 212).

$$\begin{aligned}\lambda_{121} &= \left(\sqrt{\frac{19+3\sqrt{33}}{8}} + \sqrt{\frac{11+3\sqrt{33}}{8}}\right)^6 \\ &= \left(\frac{3\sqrt{3}+\sqrt{11}}{4} + \sqrt{\frac{11+3\sqrt{33}}{8}}\right)^6.\end{aligned}$$

Proof. Letting $n = 1$ and $\Lambda = \lambda_{121}^{1/3} + \lambda_{121}^{-1/3}$ in Theorem 9.4.4, we deduce that

$$\begin{aligned}9\sqrt{3}(\Lambda^2 - \Lambda - 1)(\Lambda+2)^{1/2} - 99(\Lambda^2 - 2) + 198\sqrt{3}(\Lambda - 1)(\Lambda+2)^{1/2} \\ - 759\Lambda + 693\sqrt{3}(\Lambda+2)^{1/2} - 1386 = \Lambda(\Lambda^2 - 3).\end{aligned} \tag{9.4.12}$$

Rearranging (9.4.12), we find that

$$9\sqrt{3}(\Lambda+2)^{1/2}(\Lambda^2 + 21\Lambda + 54) = \Lambda^3 + 99\Lambda^2 + 756\Lambda + 1188,$$

and then squaring both sides, we deduce the equation

$$(\Lambda^2 - 15\Lambda - 18)^3 = 0.$$

Thus

$$\Lambda = \frac{3(5+\sqrt{33})}{2}.$$

We complete the proof by using (9.4.2) and (9.4.4). □

Lemma 9.4.1.

$$\lambda_{1/n} = \frac{1}{\lambda_n}.$$

Proof. Recall that

$$\eta(\tau+1)=e^{\pi i/12}\eta(\tau) \quad \text{and} \quad \eta(-1/\tau)=(\tau/i)^{1/2}\eta(\tau). \tag{9.4.13}$$

From these properties of the Dedekind eta function or from Entry 27(iv) in Chapter 16 of [54, p. 43], we find that

$$\eta^6\left(\frac{1+i/\sqrt{3n}}{2}\right)=\left(3n\sqrt{3n}\right)\eta^6\left(\frac{1+i\sqrt{3n}}{2}\right)$$

and

$$\eta^6\left(\frac{1+i\sqrt{3/n}}{2}\right)=\left(\frac{n}{3}\sqrt{\frac{n}{3}}\right)\eta^6\left(\frac{1+i\sqrt{n/3}}{2}\right).$$

Hence the lemma follows from (9.1.4). □

Using Theorems 9.4.2, 9.4.3, and 9.4.4, along with Lemma 9.4.1, Berndt, Chan, Kang, and Zhang [65] established the values

$$\begin{aligned}
\lambda_5&=\frac{1+\sqrt{5}}{2},\\
\lambda_7&=(2+\sqrt{3})^{3/2}\left(\frac{\sqrt{3}+\sqrt{7}}{2}\right)^{-3/2},\\
\lambda_{11}&=(2\sqrt{3}+\sqrt{11})^{3/2}(10+3\sqrt{11})^{-1/2}.
\end{aligned}$$

9.5 λ_n and Modular Equations in the Theory of Signature 3

Suppose that β has degree p over α in the theory of signature 3, and let

$$z_1 := {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\alpha\right) \qquad \text{and} \qquad z_p := {}_2F_1\left(\frac{1}{3},\frac{2}{3};1;\beta\right).$$

If $\omega := \exp(2\pi i/3)$, the cubic theta functions are defined by

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}, \tag{9.5.1}$$

$$b(q) := \sum_{m,n=-\infty}^{\infty} \omega^{m-n}q^{m^2+mn+n^2}, \tag{9.5.2}$$

and

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}. \tag{9.5.3}$$

Then [60, Lemma 2.6, Corollary 3.2]

$$a(q) = z_1, \qquad b(q) = (1-\alpha)^{1/3} z_1, \qquad c(q) = \alpha^{1/3} z_1,$$

and

$$a(q^p) = z_p, \qquad b(q^p) = (1-\beta)^{1/3} z_p, \qquad c(q^p) = \beta^{1/3} z_p.$$

Thus, it follows that

$$x := (\alpha\beta)^{1/6} = \left(\frac{c(q)c(q^p)}{a(q)a(q^p)}\right)^{1/2} \tag{9.5.4}$$

and

$$y := \{(1-\alpha)(1-\beta)\}^{1/6} = \left(\frac{b(q)b(q^p)}{a(q)a(q^p)}\right)^{1/2}. \tag{9.5.5}$$

Since [60, Lemma 5.1]

$$b(q) = \frac{f^3(-q)}{f(-q^3)}, \tag{9.5.6}$$

$$c(q) = 3q^{1/3}\frac{f^3(-q^3)}{f(-q)}, \tag{9.5.7}$$

and [90], [57, p. 96, Theorem 2.2]

$$a^3(q) = b^3(q) + c^3(q), \tag{9.5.8}$$

we find that

$$a(q) = \left\{\frac{f^{12}(-q) + 27qf^{12}(-q^3)}{f^3(-q)f^3(-q^3)}\right\}^{1/3}. \tag{9.5.9}$$

(This was also proved by Ramanujan; see [54, p. 460, Entry 3(i)].) Hence, by (9.5.4)–(9.5.7) and (9.5.9),

$$x = \frac{3q^{(p+1)/6}f^2(-q^3)f^2(-q^{3p})}{\{f^{12}(-q)+27qf^{12}(-q^3)\}^{1/6}\{f^{12}(-q^p)+27q^pf^{12}(-q^{3p})\}^{1/6}} \tag{9.5.10}$$

and

$$y = \frac{f^2(-q)f^2(-q^p)}{\{f^{12}(-q)+27qf^{12}(-q^3)\}^{1/6}\{f^{12}(-q^p)+27q^pf^{12}(-q^{3p})\}^{1/6}}. \tag{9.5.11}$$

Let

$$T := T(q) := \frac{f(-q)}{3^{1/4}q^{1/12}f(-q^3)}$$

and

$$U_p := U_p(q) := \frac{f(-q^p)}{3^{1/4}q^{p/12}f(-q^{3p})}.$$

Then from (9.5.10) and (9.5.11), we find that

$$\frac{x}{y} = (TU_p)^{-2} \tag{9.5.12}$$

and

$$xy = (T^6 + T^{-6})^{-1/3}(U_p^6 + U_p^{-6})^{-1/3}. \tag{9.5.13}$$

We now employ modular equations in x and y to calculate further values of λ_n.

Theorem 9.5.1.

$$\begin{aligned}
&32\left\{(\lambda_n\lambda_{289n})^{2/3} + (\lambda_n\lambda_{289n})^{-2/3}\right\} + 80\left\{(\lambda_n\lambda_{289n})^{1/3} + (\lambda_n\lambda_{289n})^{-1/3}\right\} \\
&\quad + 118 + (\lambda_n - \lambda_n^{-1})^{1/3}(\lambda_{289n} - \lambda_{289n}^{-1})^{1/3}\Big((\lambda_n\lambda_{289n})^{2/3} + (\lambda_n\lambda_{289n})^{-2/3} \\
&\quad + 35\left\{(\lambda_n\lambda_{289n})^{1/3} + (\lambda_n\lambda_{289n})^{-1/3}\right\} + 56\Big) \\
&\quad - (\lambda_n - \lambda_n^{-1})^{2/3}(\lambda_{289n} - \lambda_{289n}^{-1})^{2/3}\left((\lambda_n\lambda_{289n})^{1/3} + (\lambda_n\lambda_{289n})^{-1/3} - 14\right) \\
&= \frac{1}{3}\left\{\left(\frac{\lambda_n}{\lambda_{289n}}\right) + \left(\frac{\lambda_n}{\lambda_{289n}}\right)^{-1}\right\}.
\end{aligned}$$

Proof. The modular equation of degree 17 with which we begin our proof was first established by Chan and Liaw [108] and is given by

$$\begin{aligned}
x^6 &+ 96x^5y - 240x^4y^2 + 354x^3y^3 - 240x^2y^4 + 96xy^5 + y^6 - 3x^4 + 105x^3y \\
&- 168x^2y^2 + 105xy^3 - 3y^4 + 3x^2 + 42xy + 3y^2 - 1 = 0,
\end{aligned} \tag{9.5.14}$$

where x and y are defined by (9.5.4) and (9.5.5), respectively, with $p = 17$. Dividing both sides of (9.5.14) by $3x^3y^3$, we obtain

$$\begin{aligned}
&32\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) - 80\left(\frac{x}{y} + \frac{y}{x}\right) + 118 - \frac{1}{xy}\left\{\left(\frac{x^2}{y^2} + \frac{y^2}{x^2}\right) - 35\left(\frac{x}{y} + \frac{y}{x}\right) + 56\right\} \\
&\quad + \frac{1}{x^2y^2}\left\{\left(\frac{x}{y} + \frac{y}{x}\right) + 14\right\} = \frac{1}{3}\left\{\frac{1}{x^3y^3} - \left(\frac{x^3}{y^3} + \frac{y^3}{x^3}\right)\right\},
\end{aligned} \tag{9.5.15}$$

after a slight rearrangement. By (9.5.12), (9.5.13), and (9.5.15),

$$\begin{aligned}
&32\{(TU_{17})^{-4} + (TU_{17})^4\} - 80\{(TU_{17})^{-2} + (TU_{17})^2\} + 118 \\
&\quad - (T^6 + T^{-6})^{1/3}(U_{17}^6 + U_{17}^{-6})^{1/3} \\
&\quad \times \left((TU_{17})^{-4} + (TU_{17})^4 - 35\{(TU_{17})^{-2} + (TU_{17})^2\} + 56\right)
\end{aligned}$$

$$+ (T^6 + T^{-6})^{2/3}(U_{17}^6 + U_{17}^{-6})^{2/3}\left((TU_{17})^{-2} + (TU_{17})^2 + 14\right)$$
$$= \frac{1}{3}\left\{\left(\frac{T}{U_{17}}\right)^6 + \left(\frac{T}{U_{17}}\right)^{-6}\right\}. \tag{9.5.16}$$

Replacing q by $-q$ in (9.5.16), then setting $q = e^{-\pi\sqrt{n/3}}$, and lastly using (9.1.4), we complete the proof. □

Corollary 9.5.1 (p. 212).

$$\lambda_{289} = \left(\frac{\sqrt{34 + 13\sqrt[3]{17} + 5\sqrt[3]{17^2}}}{2} + \frac{\sqrt{30 + 13\sqrt[3]{17} + 5\sqrt[3]{17^2}}}{2}\right)^6.$$

Proof. Set $n = 1$ and $\lambda = \lambda_{289}$. It follows from Theorem 9.5.1 that

$$118 + 32(\lambda^{2/3} + \lambda^{-2/3}) + 80(\lambda^{1/3} + \lambda^{-1/3}) = \frac{\lambda + \lambda^{-1}}{3}. \tag{9.5.17}$$

Let $\Lambda := \lambda^{1/3} + \lambda^{-1/3}$. Then from (9.5.17), we may deduce that

$$118 + 32(\Lambda^2 - 2) + 80\Lambda = \frac{\Lambda^3 - 3\Lambda}{3}.$$

Hence Λ is a root of the equation

$$\Lambda^3 - 96\Lambda^2 - 243\Lambda - 162 = 0,$$

and the solution that we seek is given by

$$\Lambda = 32 + 13\sqrt[3]{17} + 5\sqrt[3]{17^2}.$$

Corollary 9.5.1 now follows from (9.4.2) and (9.4.4). □

Corollary 9.5.2 (p. 212).

$$\lambda_{17} = 4 + \sqrt{17}.$$

Proof. Let $n = 1/17$ in Theorem 9.5.1. Then by Lemma 9.4.1, we deduce that

$$171 - 64(\lambda_{17} - \lambda_{17}^{-1})^{2/3} + 6(\lambda_{17} - \lambda_{17}^{-1})^{4/3} = \frac{1}{6}(\lambda_{17}^2 + \lambda_{17}^{-2}).$$

We complete the proof by solving this equation for λ_{17}. □

Theorem 9.5.2 (p. 212).

$$\lambda_{169} = \left(\frac{2 + \sqrt{13}}{2} + \frac{\sqrt{13 + 4\sqrt{13}}}{2}\right)^6.$$

Proof. Let x and y be defined by (9.5.4) and (9.5.5), respectively, with $p = 13$. In his thesis [204], Liaw established a modular equation of degree 13, which, by (9.5.12) and (9.5.13), can be put in the abbreviated form

$$\begin{aligned}
&\left\{(TU_{13})^{-42} + (TU_{13})^{42}\right\} + 76142\left\{(TU_{13})^{-36} + (TU_{13})^{36}\right\} \\
&\quad + 1932468187\left\{(TU_{13})^{-30} + (TU_{13})^{30}\right\} \\
&\quad + 16346295812652\left\{(TU_{13})^{-24} + (TU_{13})^{24}\right\} \\
&\quad - 42859027901079\left\{(TU_{13})^{-18} + (TU_{13})^{18}\right\} \\
&\quad + 30681672585330\left\{(TU_{13})^{-12} + (TU_{13})^{12}\right\} \\
&\quad + 44443969755835\left\{(TU_{13})^{-6} + (TU_{13})^{6}\right\} \\
&\quad - 90882188302360 + R(x, y) = 0, \qquad (9.5.18)
\end{aligned}$$

where $R(x, y)$ contains a factor $1/(x^3y^3)$. Let $q = e^{-\pi/\sqrt{3}}$, recall that $\lambda_1 = 1$, and set $\lambda = \lambda_{169}$. Replacing q by $-q$ in (9.5.18) and using (9.1.4), we find that

$$\begin{aligned}
&-(\lambda^7 + \lambda^{-7}) + 76142(\lambda^6 + \lambda^{-6}) - 1932468187(\lambda^5 + \lambda^{-5}) \\
&\quad + 16346295812652(\lambda^4 + \lambda^{-4}) + 42859027901079(\lambda^3 + \lambda^{-3}) \\
&\quad + 30681672585330(\lambda^2 + \lambda^{-2}) - 44443969755835(\lambda + \lambda^{-1}) \\
&\quad - 90882188302360 = 0, \qquad (9.5.19)
\end{aligned}$$

since $R(x, y)$ equals 0 after q is replaced by $-q$, because

$$1/x^3y^3 = (T^6 + T^{-6})(U_{13}^6 + U_{13}^{-6})$$

is a factor of $R(x, y)$, and because

$$\begin{aligned}
&\left\{T^6(-q) + T^{-6}(-q)\right\}\left\{U_{13}^6(-q) + U_{13}^{-6}(-q)\right\} \\
&\quad = -\left\{(\lambda_1 - \lambda_1^{-1})(\lambda_{169} - \lambda_{169}^{-1})\right\}.
\end{aligned}$$

Set $\Lambda = \lambda + \lambda^{-1}$. Then (9.5.19) takes the equivalent form

$$\begin{aligned}
&-(\Lambda^7 - 7\Lambda^5 + 14\Lambda^3 - 7\Lambda) + 76142(\Lambda^6 - 6\Lambda^4 + 9\Lambda^2 - 2) \\
&\quad - 1932468187(\Lambda^5 - 5\Lambda^3 + 5\Lambda) + 16346295812652(\Lambda^4 - 4\Lambda^2 + 2) \\
&\quad + 42859027901079(\Lambda^3 - 3\Lambda) + 30681672585330(\Lambda^2 - 2) \\
&\quad - 44443969755835\Lambda - 90882188302360 = 0,
\end{aligned}$$

which simplifies to

$$(\Lambda - 2)(\Lambda^2 - 25380\Lambda - 39100)^3 = 0.$$

Therefore,

$$\lambda + \lambda^{-1} = \Lambda = 10(1269 + 352\sqrt{13}).$$

Solving the quadratic equation above, we find that

$$\lambda_{169} = 6345 + 1760\sqrt{13} + 12\sqrt{559221 + 155100\sqrt{13}}.$$

Using the formula [69, equation (3.1)]

$$\left\{(32b^3 - 6b) + \sqrt{(32b^3 - 6b)^2 - 1}\right\}^{1/6} = \sqrt{b + \frac{1}{2}} + \sqrt{b - \frac{1}{2}},$$

with $b = (15 + 4\sqrt{13})/4$, we deduce the given value of λ_{169}. $\square$

Equation (9.5.18) can also be used to deduce λ_{13}.

Theorem 9.5.3 (p. 212).

$$\lambda_{361} = \frac{1}{3}\Big(2928581 + 1097504(19)^{1/3} + 411296(19)^{2/3}$$
$$+4\sqrt{1608109304409 + 602648894772(19)^{1/3} + 225846395748(19)^{2/3}}\Big).$$

Proof. The proof is similar to that for Theorem 9.5.2. Let x and y be defined by (9.5.4) and (9.5.5), respectively, and set $u = x^3$, $v = y^3$, and $p = 19$. Liaw [204] discovered a new modular equation of degree 19, which we give in the abbreviated form

$$\begin{aligned}
\left(\frac{u^{10}}{v^{10}}\right) + \left(\frac{v^{10}}{u^{10}}\right) &- 17571484\left\{\left(\frac{u^9}{v^9}\right) + \left(\frac{v^9}{u^9}\right)\right\} \\
&+ 102919027240030\left\{\left(\frac{u^8}{v^8}\right) + \left(\frac{v^8}{u^8}\right)\right\} \\
&- 200937885610911191740\left\{\left(\frac{u^7}{v^7}\right) + \left(\frac{v^7}{u^7}\right)\right\} \\
&+ 363165905126589014509\left\{\left(\frac{u^6}{v^6}\right) + \left(\frac{v^6}{u^6}\right)\right\} \\
&- 274505067414721954283 2\left\{\left(\frac{u^5}{v^5}\right) + \left(\frac{v^5}{u^5}\right)\right\} \\
&+ 1669253999271588508904\left\{\left(\frac{u^4}{v^4}\right) + \left(\frac{v^4}{u^4}\right)\right\} \\
&- 9487507697742191502320\left\{\left(\frac{u^3}{v^3}\right) + \left(\frac{v^3}{u^3}\right)\right\} \\
&- 7070474114231105014510\left\{\left(\frac{u^2}{v^2}\right) + \left(\frac{v^2}{u^2}\right)\right\} \\
&- 7249503742499660191624\left\{\left(\frac{u}{v}\right) + \left(\frac{v}{u}\right)\right\} \\
&- 29289891786172199497868 + R(u, v) = 0,
\end{aligned}$$

where $R(u,v)$ is a sum of terms with a factor $1/(uv)$, which equals 0 after setting $q = e^{-\pi/\sqrt{3}}$ and replacing q by $-q$. With $\Lambda = \lambda_{361} + \lambda_{361}^{-1}$, we eventually find that

$$\begin{aligned}(\Lambda^{10} - 10\Lambda^8 + 35\Lambda^6 - 50\Lambda^4 + 25\Lambda^2 - 2)\\ \quad - 17571484(\Lambda^9 - 9\Lambda^7 + 27\Lambda^5 - 30\lambda^3 + 9\Lambda)\\ \quad + 102919027240030(\Lambda^8 - 8\Lambda^6 + 20\Lambda^4 - 16\Lambda^2 + 2)\\ \quad - 200937885610911191740(\Lambda^7 - 7\Lambda^5 + 14\lambda^3 - 7\Lambda)\\ \quad + 363165905126589014509(\Lambda^6 - 6\Lambda^4 + 9\Lambda^2 - 2)\\ \quad - 274505067414721954283 2(\Lambda^5 - 5\Lambda^3 + 5\Lambda)\\ \quad + 1669253999271588508904(\Lambda^4 - 4\Lambda^2 + 2)\\ \quad - 9487507697742191502320(\Lambda^3 - 3\Lambda)\\ \quad - 7070474114231105014510(\Lambda^2 - 2)\\ \quad - 7249503742499660191624\Lambda - 2928989178617219949786 8 = 0,\end{aligned}$$

which simplifies to

$$(\Lambda + 2)(-18438200 + 7433420\Lambda - 5857162\Lambda^2 + \Lambda^3)^3 = 0.$$

Hence,

$$\Lambda = \frac{1}{3}(5857162 + 2195008(19)^{1/3} + 822592(19)^{2/3}).$$

Solving the equation $\Lambda = \lambda_{361} + \lambda_{361}^{-1}$ for λ_{361}, we complete the proof. □

9.6 $\boldsymbol{\lambda_n}$ and Kronecker's Limit Formula

Let $m > 0$ be square-free and let $K = \mathbb{Q}(\sqrt{-m})$, the imaginary quadratic field with discriminant d, where

$$d = \begin{cases} -4m, & \text{if } -m \equiv 2, 3 \pmod 4, \\ -m, & \text{if } -m \equiv 1 \pmod 4. \end{cases} \tag{9.6.1}$$

Let $d = d_1 d_2$, where $d_1 > 0$ and, for $i = 1, 2$, $d_i \equiv 0$ or $1 \pmod 4$. If $\mathcal{P}$ denotes a prime ideal in K, then the Gauss genus character χ is defined by

$$\chi(\mathcal{P}) = \begin{cases} \left(\dfrac{d_1}{N(\mathcal{P})}\right), & \text{if } \ N(\mathcal{P}) \nmid d_1, \\ \left(\dfrac{d_2}{N(\mathcal{P})}\right), & \text{if } \ N(\mathcal{P}) \mid d_1, \end{cases} \tag{9.6.2}$$

where $N(\mathcal{P})$ is the norm of the ideal $\mathcal{P}$ and $\left(\frac{\cdot}{\cdot}\right)$ denotes the Kronecker symbol.

Let

$$\Omega = \begin{cases} \sqrt{-m}, & \text{if } -m \equiv 2,3 \pmod 4, \\ \dfrac{1+\sqrt{-m}}{2}, & \text{if } -m \equiv 1 \pmod 4. \end{cases} \tag{9.6.3}$$

It is known [222] that each ideal class in the class group C_K contains primitive ideals that are $\mathbb{Z}$-modules of the form $\mathcal{A} = [a, b+\Omega]$, where a and b are rational integers, $a > 0$, $a \mid N(b+\Omega)$, $|b| \le a/2$, a is the smallest positive integer in $\mathcal{A}$, and $N(\mathcal{A}) = a$. Hence, Siegel's theorem [259, p. 72], obtained from Kronecker's limit formula, can be stated as follows.

Theorem 9.6.1 (Siegel). *Let χ be a genus character arising from the decomposition $d = d_1 d_2$. Let h_i be the class number of the field $\mathbb{Q}(\sqrt{d_i})$, ω and ω_2 the numbers of roots of unity in K and $\mathbb{Q}(\sqrt{d_2})$, respectively, and ϵ_1 the fundamental unit of $\mathbb{Q}(\sqrt{d_1})$. Let*

$$F(\mathcal{A}) = \frac{|\eta(z)|^2}{\sqrt{a}}, \tag{9.6.4}$$

where $z = (b+\Omega)/a$ with $[a, b+\Omega] \in \mathcal{A}^{-1}$. Then

$$\epsilon_1^{\omega h_1 h_2/\omega_2} = \prod_{\mathcal{A}\in C_K} F(\mathcal{A})^{-\chi(\mathcal{A})}. \tag{9.6.5}$$

Ramanathan utilized Theorem 9.6.1 to compute λ_n and μ_n [236, Theorem 4].

Theorem 9.6.2 (Ramanathan). *Let $3n$ be a positive square-free integer and let $K = \mathbb{Q}(\sqrt{-3n})$ be an imaginary quadratic field such that each genus contains only one ideal class. Then*

$$\prod_{\chi} \epsilon_1^{t_\chi} = \begin{cases} \lambda_n, & \textit{if} \quad n \equiv 1 \pmod 4, \\ \mu_n, & \textit{if} \quad n \equiv 2,3 \pmod 4, \end{cases}$$

where

$$t_\chi = \frac{6\omega h_1 h_2}{\omega_2 h} \tag{9.6.6}$$

and h, h_1, h_2 are the class numbers of K, $\mathbb{Q}(\sqrt{d_1})$, and $\mathbb{Q}(\sqrt{d_2})$, respectively, ω and ω_2 are the numbers of roots of unity in K and $\mathbb{Q}(\sqrt{d_2})$, respectively, ϵ_1 is the fundamental unit in $\mathbb{Q}(\sqrt{d_1})$, and χ runs through all genus characters such that if χ corresponds to the decomposition $d_1 d_2$, then either $\left(\frac{d_1}{3}\right)$ or $\left(\frac{d_2}{3}\right) = -1$ and therefore d_1, d_2, h_1, h_2, ω_2, and ϵ_1 are dependent on χ.

With the use of Theorem 9.6.2, fourteen values of λ_n and nineteen of μ_n can be evaluated. Among them, Ramanujan recorded only the values of λ_n for which the exponent t_χ equals 1. For completeness, we state all fourteen values of such λ_n's in the following corollary, although only five of the fourteen values are indicated in Entry 9.1.1.

Corollary 9.6.1 (p. 212).

$$\lambda_5 = \frac{1+\sqrt{5}}{2}, \qquad \lambda_{17} = 4+\sqrt{17},$$

$$\lambda_{41} = 32+5\sqrt{41}, \qquad \lambda_{65} = \left(8+\sqrt{65}\right)\left(\frac{1+\sqrt{5}}{2}\right)^6,$$

$$\lambda_{89} = 500+53\sqrt{89}, \qquad \lambda_{145} = \left(\frac{1+\sqrt{5}}{2}\right)^9\left(\frac{5+\sqrt{29}}{2}\right)^3,$$

$$\lambda_{161} = \left(16\sqrt{23}+29\sqrt{7}\right)\left(\frac{3\sqrt{3}+\sqrt{23}}{2}\right)^3,$$

$$\lambda_{185} = \left(68+5\sqrt{185}\right)\left(\frac{1+\sqrt{5}}{2}\right)^{12},$$

$$\lambda_{209} = \left(46\sqrt{11}+35\sqrt{19}\right)\left(2\sqrt{3}+\sqrt{11}\right)^3,$$

$$\lambda_{265} = \left(\frac{7+\sqrt{53}}{2}\right)^3\left(\frac{1+\sqrt{5}}{2}\right)^{15},$$

$$\lambda_{385} = \left(\frac{1+\sqrt{5}}{2}\right)^9\left(\frac{5+\sqrt{21}}{2}\right)^3\left(\frac{\sqrt{7}+\sqrt{11}}{2}\right)^3\left(\frac{\sqrt{15}+\sqrt{11}}{2}\right)^3,$$

$$\lambda_{665} = \left(14\sqrt{35}+19\sqrt{19}\right)\left(2\sqrt{5}+\sqrt{21}\right)^3\left(\frac{1+\sqrt{5}}{2}\right)^{12}\left(\frac{\sqrt{15}+\sqrt{19}}{2}\right)^3,$$

$$\lambda_{1001} = \left(83\sqrt{77}+202\sqrt{13}\right)\left(2\sqrt{3}+\sqrt{11}\right)^3\left(\frac{9+\sqrt{77}}{2}\right)^3\left(\frac{7\sqrt{3}+\sqrt{143}}{2}\right)^3,$$

$$\lambda_{1105} = \left(4+\sqrt{17}\right)^3\left(8+\sqrt{65}\right)^3\left(\frac{1+\sqrt{5}}{2}\right)^{12}\left(\frac{15+\sqrt{221}}{2}\right)^3.$$

We next state a theorem that was derived in [65] from Theorem 9.6.1 and that can be used to evaluate certain λ_n for $n \equiv 3 \pmod 4$.

Theorem 9.6.3. *Let n be a positive square-free integer greater than 3, not divisible by 3, and with $n \equiv 3 \pmod 4$. Let $K = \mathbb{Q}(\sqrt{-3n})$ be an imaginary quadratic field such that each genus contains only one ideal class. Let C_0 be the principal ideal class containing $[1, \Omega]$, where Ω is defined by (9.6.3), and let C_1 and C_2 be nonprincipal ideal classes containing $[2, 1+\Omega]$ and $[6, 3+\Omega]$, respectively. Then*

$$\lambda_n = \left(\prod_{\chi(C_1)=-1} \epsilon_1^{t_\chi}\right)^{-1}\left(\prod_{\chi(C_2)=-1} \epsilon_1^{t_\chi}\right),$$

where t_χ, d_1, d_2, h_1, h_2, ω_2, and ϵ_1 are defined in Theorem 9.6.2, *and the products are over all characters χ (the first with $\chi(C_1) = -1$ and the second with $\chi(C_2) = -1$) associated with the decomposition $d = d_1 d_2$. Therefore d_1, d_2, h_1, h_2, ω_2, and ϵ_1 are dependent on χ.*

We can apply Theorem 9.6.3 to evaluate λ_n when $n = 7$, 11, 19, 31, 35, 55, 59, 91, 115, 119, 455. Since none of these values of n is on Ramanujan's list in Entry 9.1.1, we refer the reader to [65] for details.

9.7 The Remaining Five Values

At this juncture, we see that there are five values of n, namely, $n = 73, 97, 193, 217, 241$ in Ramanujan's Entry 9.1.1, for which we have not calculated the associated value of λ_n. In [65], we used an empirical process to derive these values of λ_n. This empirical process is analogous to those used by Watson [275], [276] in his computations of the Ramanujan–Weber class invariants G_n and g_n.

Theorem 9.7.1 (p. 212). *We have*

$$\lambda_{73} = \left(\sqrt{\frac{11+\sqrt{73}}{8}} + \sqrt{\frac{3+\sqrt{73}}{8}}\right)^6, \tag{9.7.1}$$

$$\lambda_{97} = \left(\sqrt{\frac{17+\sqrt{97}}{8}} + \sqrt{\frac{9+\sqrt{97}}{8}}\right)^6, \tag{9.7.2}$$

$$\lambda_{241} = \left(16+\sqrt{241} + \sqrt{496+32\sqrt{241}}\right)^3, \tag{9.7.3}$$

$$\lambda_{217} = \left(\sqrt{\frac{1901+129\sqrt{217}}{8}} + \sqrt{\frac{1893+129\sqrt{217}}{8}}\right)^{3/2} \tag{9.7.4}$$

$$\times \left(\sqrt{\frac{1597+108\sqrt{217}}{4}} + \sqrt{\frac{1593+108\sqrt{217}}{4}}\right)^{3/2}, \tag{9.7.5}$$

$$\lambda_{193}^{1/3} + \frac{1}{\lambda_{193}^{1/3}} = \frac{1}{4}\left(39+3\sqrt{193} + \sqrt{2690+194\sqrt{193}}\right). \tag{9.7.6}$$

Chan, Gee, and Tan [107] employed more sophisticated methods including the Shimura reciprocity law to give rigorous proofs of the evaluations of λ_{73} and λ_{217}. In the next three sections, we will follow the method in [107] and give a complete proof of Theorem 9.7.1 .

9.8 Some Modular Functions of Level 72

For $\tau \in \mathbb{C}$ and $\operatorname{Im}\tau > 0$, define

$$\mathfrak{g}_0(\tau) := \frac{\eta\left(\frac{\tau}{3}\right)}{\eta(\tau)},$$

where the Dedekind eta function $\eta(\tau)$ is defined by (9.1.3). The function $\mathfrak{g}_0(\tau)$ is a modular function of level 72. This means that it is meromorphic on the completed upper half-plane $\mathbb{H}\cup\mathbb{Q}\cup\{\infty\}$, admits a Laurent series expansion in the variable $q^{1/72} = e^{2\pi i\tau/72}$ centered at $q = 0$, and is invariant with respect to the matrix group

$$\Gamma(72) := \ker[\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/72\mathbb{Z})].$$

It is known that the minimal polynomial for $\mathfrak{g}_0^{12}(\tau)$ over the modular function field $\mathbb{Q}(j(\tau))$, with $j(\tau)$ the modular j-invariant defined by (9.2.1), is given by [107, (2.5)]

$$X^4 + 36\,X^3 + 270\,X^2 + (756 - j(\tau))X + 3^6. \tag{9.8.1}$$

Over $\mathbb{Q}(j(\tau))$, the other roots of (9.8.1) are $\mathfrak{g}_1^{12}(\tau)$, $\mathfrak{g}_2^{12}(\tau)$, and $\mathfrak{g}_3^{12}(\tau)$ defined by

$$\mathfrak{g}_1(\tau) := \zeta_{24}^{-1}\frac{\eta((\tau+1)/3)}{\eta(\tau)}, \quad \mathfrak{g}_2(\tau) := \frac{\eta((\tau+2)/3)}{\eta(\tau)}, \quad \text{and} \quad \mathfrak{g}_3(\tau) := \sqrt{3}\frac{\eta(3\tau)}{\eta(\tau)}. \tag{9.8.2}$$

The proof of the following theorem is similar to the proof of [107, Theorem 2.1].

Theorem 9.8.1. *If n is square-free and $n \equiv 1 \pmod 4$ then*

$$\mathfrak{g}_2^{12}\left(\frac{-1+\sqrt{-3n}}{2}\right)$$

lies in the Hilbert class field K_1 associated with $K = \mathbb{Q}(\sqrt{-3n})$.

If K is an imaginary quadratic field of discriminant D, then, by class field theory, there exists an isomorphism

$$\mathrm{Gal}(K_1/K) \simeq C(D) \tag{9.8.3}$$

between the Galois group for $K \subset K_1$ and the form class group of discriminant D. Among the primitive quadratic forms $[a,b,c]$ having discriminant $D = b^2 - 4ac$, we can find a complete set of representatives in $C(D)$ by choosing the reduced forms

$$|\,b\,| \le a \le c \quad \text{and} \quad b \ge 0, \qquad \text{if either } |\,b\,| = a \text{ or } a = c\,.$$

In view of (9.8.3), it is therefore not surprising that we can define an action of $C(D)$ on certain elements arising from modular functions that lie in K_1. The main result is the following explicit form of the Shimura reciprocity law.

Lemma 9.8.1. *Let K be an imaginary quadratic number field of odd discriminant D, and let $h \in F_m$, where F_m is the field of modular functions of level m over $\mathbb{Q}(\zeta_m)$ with $h((-1+\sqrt{D})/2) \in K_1$. Given a primitive quadratic form $[a,b,c]$ of discriminant D, let $M = M_{[a,b,c]} \in \mathrm{GL}_2(\mathbb{Z}/m\mathbb{Z})$ be the matrix that satisfies the congruences*

$$M \equiv \begin{cases} \begin{pmatrix} a & \frac{b-1}{2} \\ 0 & 1 \end{pmatrix} \pmod{p^r}, & \text{if } p \nmid a, \\[2ex] \begin{pmatrix} \frac{-b-1}{2} & -c \\ 1 & 0 \end{pmatrix} \pmod{p^r}, & \text{if } p \mid a \text{ and } p \nmid c, \\[2ex] \begin{pmatrix} \frac{-b-1}{2} - a & \frac{1-b}{2} - c \\ 1 & -1 \end{pmatrix} \pmod{p^r}, & \text{if } p \mid a \text{ and } p \mid c, \end{cases}$$

at all prime power factors $p^r \mid m$. The Galois action of the class of $[a,-b,c]$ in $C(D)$ with respect to the Artin map is given by

$$\left(h\left(\frac{-1+\sqrt{D}}{2}\right)\right)^{[a,-b,c]} = h^M\left(\frac{-b+\sqrt{D}}{2a}\right),$$

where h^M denotes the image of h under the action of M.

For a proof of Lemma 9.8.1, see Gee's paper [153].

We apply Lemma 9.8.1 with $h = \mathfrak{g}_2^{12}$, $D = -3n$ for $n \equiv 1 \pmod 4$, and $m = 72$. It remains to determine the action of M on $\mathfrak{g}_2^{12}$.

First, note that the action of such an M depends only on $M_{p^{r_p}}$ for all prime factors $p \mid m$, where $M_N \in \mathrm{GL}_2(\mathbb{Z}/N\mathbb{Z})$ is the reduction modulo N of M, and r_p is the largest power of p such that p^{r_p} divides m.

Now, every M_N with determinant x decomposes as

$$M_N = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_N \begin{pmatrix} a & b \\ c & d \end{pmatrix}_N$$

for some $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)_N \in \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$. Since $\mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is generated by S_N and T_N, where $S = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, it suffices to find the action of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right)_{p^{r_p}}$, $S_{p^{r_p}}$, and $T_{p^{r_p}}$ on h for all $p \mid m$.

For $\left(\begin{smallmatrix} 1 & 0 \\ 0 & x \end{smallmatrix}\right)_{p^{r_p}}$, the action on F_m is given by lifting the automorphism of $\mathbb{Q}(\zeta_m)$ determined by

$$\zeta_{p^{r_p}} \mapsto \zeta_{p^{r_p}}^x \qquad \text{and} \qquad \zeta_{q^{r_q}} \mapsto \zeta_{q^{r_q}},$$

for all prime factors $q \mid m$ such that $q \neq p$.

In order that the actions of the matrices at different primes commute with each other, we need to lift $S_{p^{r_p}}$ and $T_{p^{r_p}}$ to matrices in $\mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})$ such that they reduce to the identity matrix in $\mathrm{SL}_2(\mathbb{Z}/q^{r_q}\mathbb{Z})$ for all $q \neq p$. In our case, for $m = 72$, the prime powers are 8 and 9, and

$$S_8 \mapsto \begin{pmatrix} -8 & 9 \\ -9 & -8 \end{pmatrix}_{72}, \qquad T_8 \mapsto \begin{pmatrix} 1 & 9 \\ 0 & 1 \end{pmatrix}_{72},$$
$$S_9 \mapsto \begin{pmatrix} 9 & -8 \\ 8 & 9 \end{pmatrix}_{72}, \qquad T_9 \mapsto \begin{pmatrix} 1 & -8 \\ 0 & 1 \end{pmatrix}_{72}.$$

When $h \in F_m$ is an η-quotient, we can use the transformation rules

$$\eta \circ S_m(\tau) = \sqrt{-i\tau}\,\eta(\tau) \qquad \text{and} \qquad \eta \circ T_m(\tau) = \zeta_{24}\eta(\tau)$$

to determine the action of any $M_m \in \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z})$. In particular,

$$(\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3) \circ S_{72} = (\mathfrak{g}_3, \zeta_{24}^{10}\mathfrak{g}_2, \zeta_{24}^{14}\mathfrak{g}_1, \mathfrak{g}_0)$$

and

$$(\mathfrak{g}_0, \mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_3) \circ T_{72} = (\mathfrak{g}_1, \zeta_{24}^{-2}\mathfrak{g}_2, \mathfrak{g}_0, \zeta_{24}^{2}\mathfrak{g}_3).$$

Consequently, we derive the following actions:

	$\mathfrak{g}_0^{12}$	$\mathfrak{g}_1^{12}$	$\mathfrak{g}_2^{12}$	$\mathfrak{g}_3^{12}$
$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_8$	$\mathfrak{g}_0^{12}$	$\mathfrak{g}_1^{12}$	$\mathfrak{g}_2^{12}$	$\mathfrak{g}_3^{12}$
S_8	$\mathfrak{g}_0^{12}$	$\mathfrak{g}_1^{12}$	$\mathfrak{g}_2^{12}$	$\mathfrak{g}_3^{12}$
T_8	$\mathfrak{g}_0^{12}$	$\mathfrak{g}_1^{12}$	$\mathfrak{g}_2^{12}$	$\mathfrak{g}_3^{12}$
$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_9, 3\mid(x-1)$	$\mathfrak{g}_0^{12}$	$\mathfrak{g}_1^{12}$	$\mathfrak{g}_2^{12}$	$\mathfrak{g}_3^{12}$
$\begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}_9, 3\mid(x-2)$	$\mathfrak{g}_0^{12}$	$\mathfrak{g}_2^{12}$	$\mathfrak{g}_1^{12}$	$\mathfrak{g}_3^{12}$
S_9	$\mathfrak{g}_3^{12}$	$\mathfrak{g}_2^{12}$	$\mathfrak{g}_1^{12}$	$\mathfrak{g}_0^{12}$
T_9	$\mathfrak{g}_1^{12}$	$\mathfrak{g}_2^{12}$	$\mathfrak{g}_0^{12}$	$\mathfrak{g}_3^{12}$

We prove the following theorem using the table above.

Theorem 9.8.2. *The action of a reduced primitive quadratic form* $[a, b, c]$ *with discriminant* D *in* $C(D)$ *on* $\mathfrak{g}_2^{12}((-1+\sqrt{D})/2)$ *is given by*

$$\left\{\mathfrak{g}_2\left(\frac{-1+\sqrt{D}}{2}\right)^{12}\right\}^{[a,-b,c]} = \begin{cases} \mathfrak{g}_0\left(\dfrac{-b+\sqrt{D}}{2a}\right)^{12}, & \text{if } b \equiv 0, a \not\equiv 0 \pmod 3, \\ \mathfrak{g}_1\left(\dfrac{-b+\sqrt{D}}{2a}\right)^{12}, & \text{if } ab \equiv -1 \pmod 3, \\ \mathfrak{g}_2\left(\dfrac{-b+\sqrt{D}}{2a}\right)^{12}, & \text{if } ab \equiv 1 \pmod 3, \\ \mathfrak{g}_3\left(\dfrac{-b+\sqrt{D}}{2a}\right)^{12}, & \text{if } a \equiv 0 \pmod 3. \end{cases}$$

Proof. We first observe that the action of M_8 on $\mathfrak{g}_0^{12}$ is trivial. Therefore, it suffices to consider the action of M_9 on $\mathfrak{g}_0^{12}$. When $3 \nmid a$,

$$M_9 = \begin{pmatrix} a & \frac{b-1}{2} \\ 0 & 1 \end{pmatrix} \equiv \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{b-1}{2a} \\ 0 & 1 \end{pmatrix} \equiv S_9 \begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix} S_9 T_9^{(b-1)/(2a)}.$$

If $b \equiv 0 \pmod 3$ and $a \equiv 1 \pmod 3$, then

$$\begin{aligned} (\mathfrak{g}_2^{12})^{M_9} &= (\mathfrak{g}_2^{12})^{S_9 \left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right) S_9 T_9^{(b-1)/(2a)}} = (\mathfrak{g}_1^{12})^{\left(\begin{smallmatrix} 1 & 0 \\ 0 & a \end{smallmatrix}\right) S_9 T_9^{(b-1)/(2a)}} \\ &= (\mathfrak{g}_1^{12})^{S_9 T_9^{(b-1)/(2a)}} = (\mathfrak{g}_2^{12})^{T_9^{(b-1)/(2a)}} = \mathfrak{g}_0^{12}, \end{aligned} \tag{9.8.4}$$

since

$$\frac{b-1}{2a} \equiv 1 \pmod 3,$$

when $b \equiv 0 \pmod 3$ and $a \equiv 1 \pmod 3$. Similarly, when $b \equiv 0 \pmod 3$ and $a \equiv 2 \pmod 3$, we see that

$$(\mathfrak{g}_2^{12})^{M_9} = \mathfrak{g}_0^{12}.$$

This proves the first case of Theorem 9.8.2.

For the second case, note that $ab \equiv -1 \pmod 3$. If $a \equiv 1 \pmod 3$ and $b \equiv -1 \pmod 3$, then the computations are the same as in (9.8.4) except for the final step, namely,

$$(\mathfrak{g}_2^{12})^{M_9} = (\mathfrak{g}_2^{12})^{T_9^{(b-1)/(2a)}} = \mathfrak{g}_1^{12},$$

since

$$\frac{b-1}{2a} \equiv 2 \pmod 3.$$

The remainder of the cases can be established in a similar way. In the last case, since $3 \mid a$, we see that, by Lemma 9.8.1,

$$\begin{aligned} M_9 &= \begin{pmatrix} \frac{-b-1}{2} & -c \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} c & \frac{-b-1}{2} \\ 0 & 1 \end{pmatrix} S_9 \\ &\equiv \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & \frac{-b-1}{2c} \\ 0 & 1 \end{pmatrix} S_9 \equiv S_9 \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} S_9 T_9^{(-b-1)/(2c)} S_9. \end{aligned}$$

□

9.9 Computations of λ_n Using the Shimura Reciprocity Law

We first note that

$$\lambda_n^2 = -\frac{1}{27} \mathfrak{g}_2^{12} \left(\frac{-1+\sqrt{-3n}}{2} \right).$$

This identification allows us to compute λ_n using Theorem 9.8.2.

Proof of Theorem 9.7.1. For $p = 73, 97$, and 241, all of which are primes, the class groups corresponding to these primes are of the form $\mathbb{Z}_4$ and the computations for all these values are similar. We will discuss only the computations of λ_{73} in detail.

Set

$$P_p = \lambda_p^2 + \frac{1}{\lambda_p^2}. \tag{9.9.1}$$

Note that λ_n^2 is a real unit [199, p. 166, Corollary], and hence P_p is an algebraic integer.

The class group of $\mathbb{Q}(\sqrt{-219})$ is generated by the form $[5, 1, 11]$. Consider the expressions $P_{73} + P_{73}^{[5,1,11]}$ and $P_{73}P_{73}^{[5,1,11]}$, where $P_{73}^{[5,1,11]}$ denotes the image of P_{73} under the action of $[5, 1, 11]$. These are both fixed by all the elements in the class group $C(\sqrt{-219})$, and hence they must be rational. On the other hand, they are algebraic integers, and this implies that they must be integers.

By Theorem 9.8.2, we can then determine that

$$P_{73} + P_{73}^{[5,1,11]} = 199044$$

and

$$P_{73}P_{73}^{[5,1,11]} = 287492.$$

Hence, P_{73} satisfies the quadratic polynomial

$$x^2 - 199044x + 287492 = 0.$$

Solving this equation and simplifying, we deduce that

$$\lambda_{73}^2 = \sqrt{C} + \sqrt{C-1},$$

where

$$C = 4952242369 + 579616128\sqrt{73}. \tag{9.9.2}$$

It turns out that the right-hand side of (9.9.2) can be written as a power of an element in $\mathbb{Q}(\sqrt{73})$. In this case we find that if $B = ((11 + \sqrt{73})/8)$, then

$$(\sqrt{B} + \sqrt{B-1})^{12} = \sqrt{C} + \sqrt{C-1}.$$

Hence we may conclude that

$$\lambda_{73} = \left(\sqrt{\frac{11+\sqrt{73}}{8}} + \sqrt{\frac{3+\sqrt{73}}{8}}\right)^6,$$

which is (9.7.1).

The calculations for $n = 97$ and 241 are similar.

Recall that P_p is defined by (9.9.1). When $p = 97$, the class group of $\mathbb{Q}(\sqrt{-291})$ is generated by $[5, 3, 15]$. Therefore,

$$P_{97} + P_{97}^{[5,3,15]} = 2122308$$

and

$$P_{97} P_{97}^{[5,3,15]} = -2833276.$$

Solving these last two equations simultaneously, we conclude that

$$P_{97} = 1061154 + 107744\sqrt{97}.$$

Using this value in (9.9.1), we solve for λ_{97} to deduce (9.7.2).

The class group of $\mathbb{Q}(\sqrt{-723})$ is generated by $[11, -5, 17]$. We therefore find that

$$P_{241} + P_{241}^{[11,-5,17]} = 62717184900$$

and

$$P_{241} P_{241}^{[11,-5,17]} = 123706369796.$$

Hence,

$$P_{241} = 31358592450 + 2019984512\sqrt{241}.$$

Using (9.9.1), we can now complete the calculation of λ_{241}.

We now turn to the case $n = 217$.

Here 217 is divisible by two primes, namely, 7 and 31. In this case, we consider two numbers Q_{217} and R_{217} defined by

$$Q_{217} = \lambda_{217}^2 \lambda_{31/7}^2 + \frac{1}{\lambda_{217}^2 \lambda_{31/7}^2}$$

and

$$R_{217} = \frac{\lambda_{217}^2}{\lambda_{31/7}^2} + \frac{\lambda_{31/7}^2}{\lambda_{217}^2}.$$

Note that the class group of $\mathbb{Q}(\sqrt{-651})$ is generated by $a := [5, 3, 33]$ and $b := [3, 3, 55]$. The order of a is 4, and the group generated by a^2 and b fixes Q_{217} and R_{217}. Hence it suffices to determine the action of b on Q_{217} and R_{217}, which can easily be done by Theorem 9.8.2. The value of λ_{217} resulting from these considerations is a product of two units given by

$$\begin{aligned}\lambda_{217} = &\left(\sqrt{\frac{1901 + 129\sqrt{217}}{8}} + \sqrt{\frac{1893 + 129\sqrt{217}}{8}}\right)^{3/2} \\ &\times \left(\sqrt{\frac{1597 + 108\sqrt{217}}{4}} + \sqrt{\frac{1593 + 108\sqrt{217}}{4}}\right)^{3/2}.\end{aligned}$$

Finally, consider the case $n = 193$. Here the class group of $\mathbb{Q}(\sqrt{-579})$ is generated by $a := [5, 1, 29]$, and it is of order 8. We consider the expression P_{193}, where P_p is given by (9.9.1). To determine P_{193}, we compute the image of P_{193} under a, a^2, and a^3. Our computations show that if

$$\begin{aligned}
\alpha &:= P_{193}, \\
\beta &:= P_{193}^{a} = -\frac{1}{27}\mathfrak{g}_2^{12}\left(\frac{1+\sqrt{-579}}{10}\right) - 27\mathfrak{g}_2^{-12}\left(\frac{1+\sqrt{-579}}{10}\right), \\
\gamma &:= P_{193}^{a^2} = -\frac{1}{27}\mathfrak{g}_0^{12}\left(\frac{3+\sqrt{-579}}{14}\right) - 27\mathfrak{g}_0^{-12}\left(\frac{3+\sqrt{-579}}{14}\right),
\end{aligned}$$

and

$$\delta := P_{193}^{a^3} = -\frac{1}{27}\mathfrak{g}_0^{12}\left(\frac{-9+\sqrt{-579}}{22}\right) - 27\mathfrak{g}_0^{-12}\left(\frac{-9+\sqrt{-579}}{22}\right),$$

then

$$\begin{aligned}
\alpha+\beta+\gamma+\delta &= 3251132424, \\
\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta &= 82707128352, \\
\alpha\beta\gamma+\alpha\beta\delta+\beta\gamma\delta+\alpha\gamma\delta &= 9465475096,
\end{aligned}$$

and

$$\alpha\beta\gamma\delta = 176664526832.$$

Solving the quartic polynomial satisfied by P_{193} and simplifying, we deduce that

$$\lambda_{193}^{1/3} + \frac{1}{\lambda_{193}^{1/3}} = \frac{1}{4}\left(39 + 3\sqrt{193} + \sqrt{2690 + 194\sqrt{193}}\right).$$

$\square$

10

Miscellaneous Results on Elliptic Functions and Theta Functions

10.1 A Quasi-theta Product

At the top of page 209 of his lost notebook [244], Ramanujan recorded the following enigmatic formula.

Entry 10.1.1 (p. 209).

$$\left\{\prod_{n=0}^{\infty}\left(\frac{1-(-1)^n q^{(2n+1)/2}}{1+(-1)^n q^{(2n+1)/2}}\right)^{2n+1}\right\}^{\log q}\left\{\prod_{n=1}^{\infty}\left(\frac{1+(-1)^n iq'^n}{1-(-1)^n iq'^n}\right)^{n}\right\}^{2\pi i}$$
$$= \exp\left(\frac{\pi^2}{4} - \frac{k\,{}_3F_2(1,1,1;\frac{3}{2},\frac{3}{2};k^2)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)}\right), \tag{10.1.1}$$

where

$$q = \exp(-\pi K'/K), \qquad q' = \exp(-\pi K/K'), \quad \textit{and} \quad 0 < k < 1. \tag{10.1.2}$$

Because of poor photocopying, (10.1.1) is very difficult to read in [244]. If the powers $2n+1$ and n on the two pairs of large parentheses were absent, the products could be expressed in terms of theta functions. Ramanujan did not use the notation ${}_3F_2$ and ${}_2F_1$ for hypergeometric functions, but instead only recorded the first three terms of each series. Also, Ramanujan did not divulge the meaning of the notations K and K'. However, from considerable work in both the ordinary notebooks [243] and lost notebook [244], we can easily deduce that K denotes the complete elliptic integral of the first kind defined by

$$K := K(k) := \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1-k^2\sin^2\varphi}},$$

where k, $0 < k < 1$, denotes the *modulus*. Furthermore, $K' = K(k')$, where $k' := \sqrt{1-k^2}$ is the *complementary modulus.*

There are no other formulas like (10.1.1) in Ramanujan's work, and apparently there are none like it in the literature as well. The purpose of the

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first few sections in this chapter is to prove (10.1.1). As will be seen in our proof, the unique character of (10.1.1) derives from a single, almost miraculous, connection with the theory of elliptic functions given in the identity

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \cosh\{(2n+1)\alpha/2\}} = \frac{k}{2z}\ {}_3F_2(1,1,1;\tfrac{3}{2},\tfrac{3}{2};k^2), \qquad (10.1.3)$$

where $\alpha = \pi K'/K$ and $z = {}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)$. The identity (10.1.3) is found in Entry 6 of Chapter 18 in Ramanujan's second notebook [243], [54, p. 153]. Like many of Ramanujan's discoveries, (10.1.3) is not fully understood. Is this connection between hypergeometric series and elliptic functions a singular accident, or are there deeper, still to be recognized connections? In his notebooks [243, p. 280], Ramanujan also attempted to find a formula similar to (10.1.3), but with $(2n+1)^2$ replaced by $(2n+1)^4$. In fact, Ramanujan struck out his imprecisely stated formula by putting two lines through it. See [57, pp. 397–403] for Berndt's failed attempt to find a correct version.

W. Duke [147] has significantly added to our understanding of Entry 10.1.1, Entry 6 of Chapter 18 in the second notebook, and the purged entry on page 280 of the second notebook [57, pp. 402–403, Entry 78]. Duke points out that the series in Entry 6 and on page 280 can be regarded as Eisenstein series of negative weight. Moreover, he establishes a corrected version of the aforementioned Entry 78. Duke's proof of Entry 6 (and a considerable generalization) relies on a double integral of hypergeometric functions given as Entry 31(ii) in Chapter 11 in Ramanujan's second notebook [243], [53, p. 88], for which he gives a shorter, more elegant proof than that given in [53, pp. 89–92]. Besides Eisenstein series of negative weight being represented by hypergeometric series, Duke gives further examples of cusp forms also represented by hypergeometric series.

In Section 10.2, we first establish in Theorem 10.2.1 an equivalent formulation of (10.1.1) as an identity among infinite series of hyperbolic trigonometric functions. Secondly, we prove this identity.

In Section 10.3, we briefly indicate generalizations of (10.1.1) and Theorem 10.2.1 and offer some related hyperbolic series of Ramanujan.

The content of the present and following two sections is taken from a paper by Berndt, H.H. Chan, and A. Zaharescu [68].

10.2 An Equivalent Formulation of (10.1.1) in Terms of Hyperbolic Series

Theorem 10.2.1. *Let α and β be any complex numbers with nonzero real parts and with $\alpha\beta = \pi^2$. Then* (10.1.1) *is equivalent to the identity*

$$\alpha \sum_{n=0}^{\infty} \frac{\sinh\{(2n+1)\alpha/2\}}{(2n+1)\cosh^2\{(2n+1)\alpha/2\}} + \pi \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\cosh^2\{(2n+1)\beta/2\}}$$

$$= \frac{\pi^2}{4} - 2\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2\cosh\{(2n+1)\alpha/2\}}. \tag{10.2.1}$$

Proof. We assume that α and β are positive real numbers. The general result will then follow by analytic continuation. Taking logarithms on both sides of (10.1.1), we find that

$$\log\left(\left\{\prod_{n=0}^{\infty}\left(\frac{1-(-1)^n q^{(2n+1)/2}}{1+(-1)^n q^{(2n+1)/2}}\right)^{2n+1}\right\}^{\log q}\right)$$
$$+\log\left(\left\{\prod_{n=1}^{\infty}\left(\frac{1+(-1)^n i q'^n}{1-(-1)^n i q'^n}\right)^{n}\right\}^{2\pi i}\right) = \frac{\pi^2}{4} - \frac{k\ {}_3F_2(1,1,1;\frac{3}{2},\frac{3}{2};k^2)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)}. \tag{10.2.2}$$

(Here and in the following step, we have ignored branches of the logarithm. The justification lies in our eventual proof of (10.2.1).) For brevity, let L and R denote, respectively, the left and right sides of (10.2.2). Then

$$L = \log q\left(\sum_{n=0}^{\infty}(2n+1)\left\{\log\left(1-(-1)^n q^{(2n+1)/2}\right)\right.\right.$$
$$\left.\left. - \log\left(1+(-1)^n q^{(2n+1)/2}\right)\right\}\right)$$
$$+ 2\pi i\left(\sum_{n=1}^{\infty} n\left\{\log\left(1+(-1)^n i q'^n\right) - \log\left(1-(-1)^n i q'^n\right)\right\}\right)$$
$$=: \log q(S_1 - S_2) + 2\pi i(S_3 - S_4). \tag{10.2.3}$$

Recall that q and q' are defined in (10.1.2). Set $\alpha = \pi K'/K$ and $\beta = \pi K/K'$, so that $\alpha\beta = \pi^2$. We now proceed to show that $S_1, \dots, S_4$ can be expressed as sums of hyperbolic functions.

Using the Taylor series of $\log(1+z)$ about $z=0$ and recalling the definition of β, we find that

$$S_3 = -\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} n\frac{(-1)^{m+mn} i^m e^{-\beta mn}}{m}$$
$$= -\sum_{m=1}^{\infty}\frac{(-i)^m}{m}\sum_{n=1}^{\infty} n\left\{(-e^{-\beta})^m\right\}^n$$
$$= -\sum_{m=1}^{\infty}\frac{(ie^{-\beta})^m}{m(1-(-e^{-\beta})^m)^2}. \tag{10.2.4}$$

By a similar calculation,

$$S_4 = -\sum_{m=1}^{\infty} \frac{(-ie^{-\beta})^m}{m(1-(-e^{-\beta})^m)^2}. \tag{10.2.5}$$

Combining (10.2.4) and (10.2.5), we find that

$$\begin{aligned}
S_3 - S_4 &= \sum_{m=1}^{\infty} \frac{-(ie^{-\beta})^m + (-ie^{-\beta})^m}{m(1-(-e^{-\beta})^m)^2} \\
&= -2i \sum_{m=0}^{\infty} \frac{(-1)^m e^{-(2m+1)\beta}}{(2m+1)(1+e^{-(2m+1)\beta})^2} \\
&= -\frac{i}{2} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)\cosh^2\{(2m+1)\beta/2\}}.
\end{aligned} \tag{10.2.6}$$

Next, again using the Taylor series of $\log(1+z)$ about $z=0$ and recalling the definition of α, we find that

$$\begin{aligned}
S_1 &= -\sum_{n=0}^{\infty}\sum_{m=1}^{\infty} (2n+1)\frac{(-1)^{mn} e^{-\alpha(2n+1)m/2}}{m} \\
&= -\sum_{m=1}^{\infty} \frac{1}{m}\left(\sum_{n=0}^{\infty}\left(2n(-1)^{mn}e^{-\alpha(2n+1)m/2} + (-1)^{mn}e^{-\alpha(2n+1)m/2}\right)\right) \\
&= -\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{2(-1)^m e^{-3\alpha m/2}}{(1-(-e^{-\alpha})^m)^2} + \frac{e^{-\alpha m/2}}{1-(-e^{-\alpha})^m}\right) \\
&= -\sum_{m=1}^{\infty} \frac{(-1)^m e^{-3\alpha m/2} + e^{-\alpha m/2}}{m(1-(-e^{-\alpha})^m)^2}.
\end{aligned} \tag{10.2.7}$$

By an analogous argument,

$$S_2 = -\sum_{m=1}^{\infty} \frac{e^{-3\alpha m/2} + (-1)^m e^{-\alpha m/2}}{m(1-(-e^{-\alpha})^m)^2}. \tag{10.2.8}$$

Thus, combining (10.2.7) and (10.2.8), we deduce that

$$\begin{aligned}
S_1 - S_2 &= \sum_{m=1}^{\infty} \frac{-(-1)^m e^{-3\alpha m/2} - e^{-\alpha m/2} + e^{-3\alpha m/2} + (-1)^m e^{-\alpha m/2}}{m(1-(-e^{-\alpha})^m)^2} \\
&= 2\sum_{m=0}^{\infty} \frac{e^{-3(2m+1)\alpha/2} - e^{-(2m+1)\alpha/2}}{(2m+1)(1+e^{-(2m+1)\alpha})^2} \\
&= 2\sum_{m=0}^{\infty} \frac{e^{-(2m+1)\alpha/2} - e^{(2m+1)\alpha/2}}{(2m+1)(e^{(2m+1)\alpha/2} + e^{-(2m+1)\alpha/2})^2} \\
&= -\sum_{m=0}^{\infty} \frac{\sinh\{(2m+1)\alpha/2\}}{(2m+1)\cosh^2\{(2m+1)\alpha/2\}}.
\end{aligned} \tag{10.2.9}$$

If we use (10.2.6) and (10.2.9) in (10.2.3) and recall that $\log q = -\alpha$, we deduce that

$$\alpha \sum_{m=0}^{\infty} \frac{\sinh\{(2m+1)\alpha/2\}}{(2m+1)\cosh^2\{(2m+1)\alpha/2\}} + \pi \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)\cosh^2\{(2m+1)\beta/2\}}$$
$$= \frac{\pi^2}{4} - \frac{k\, {}_3F_2(1,1,1;\frac{3}{2},\frac{3}{2};k^2)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)}. \tag{10.2.10}$$

We now invoke (10.1.3). If we substitute (10.1.3) into (10.2.10), we deduce (10.2.1) to complete the proof. □

It should be emphasized that the only time we used the definitions (10.1.2) of q and q' in our proof is in the application of (10.1.3). Thus, it would seem that (10.1.1) is a very special result in that there are likely very few (if any) other results like it.

We now prove (10.2.1).

Proof of (10.2.1). Our first main idea is to introduce the functions F and G in (10.2.11) and (10.2.13), respectively, and use them to find a simpler identity that is equivalent to (10.2.1). Define

$$F(\alpha) := \frac{1}{\alpha} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \cosh\{(2n+1)\alpha/2\}}. \tag{10.2.11}$$

Then

$$F'(\alpha) := -\frac{1}{2\alpha^2} \left(\alpha \sum_{n=0}^{\infty} \frac{\sinh\{(2n+1)\alpha/2\}}{(2n+1)\cosh^2\{(2n+1)\alpha/2\}} + 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \cosh\{(2n+1)\alpha/2\}} \right). \tag{10.2.12}$$

Set

$$G(\beta) := 2\beta \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \cosh\{(2n+1)\pi^2/(2\beta)\}} = 2\pi^2 F\left(\frac{\pi^2}{\beta}\right), \tag{10.2.13}$$

by (10.2.11) and the fact that $\alpha\beta = \pi^2$. Thus, by (10.2.12),

$$G'(\beta) = 2\pi^2 F'\left(\frac{\pi^2}{\beta}\right)\left(-\frac{\pi^2}{\beta^2}\right) = \alpha \sum_{n=0}^{\infty} \frac{\sinh\{(2n+1)\pi^2/(2\beta)\}}{(2n+1)\cosh^2\{(2n+1)\pi^2/(2\beta)\}} + 2 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 \cosh\{(2n+1)\pi^2/(2\beta)\}}. \tag{10.2.14}$$

If we define

$$H(\beta) := \frac{\beta\pi^2}{4} - 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n \tanh\{(2n+1)\beta/2\}}{(2n+1)^2}, \tag{10.2.15}$$

then

$$H'(\beta) := \frac{\pi^2}{4} - \pi \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\cosh^2\{(2n+1)\beta/2\}}. \tag{10.2.16}$$

In view of (10.2.14) and (10.2.16), we see that (10.2.1) is equivalent to

$$G'(\beta) = H'(\beta).$$

It follows that for some constant c,

$$G(\beta) = H(\beta) + c. \tag{10.2.17}$$

Clearly, from the definitions of $G(\beta)$ and $H(\beta)$ in (10.2.13) and (10.2.15), respectively, both $G(\beta)$ and $H(\beta)$ tend to 0 as $\beta \to 0$. Thus, in (10.2.17), $c = 0$.

Hence, it now suffices to prove that

$$\begin{aligned} 2\beta \sum_{n=0}^{\infty} &\frac{1}{(2n+1)^2 \cosh\{(2n+1)\pi^2/(2\beta)\}} \\ &= \frac{\beta\pi^2}{4} - 2\pi \sum_{n=0}^{\infty} \frac{(-1)^n \tanh\{(2n+1)\beta/2\}}{(2n+1)^2}. \end{aligned} \tag{10.2.18}$$

It is easily seen that (10.2.18) is equivalent to

$$\begin{aligned} \beta \sum_{n=-\infty}^{\infty} &\frac{1}{(2n+1)^2 \cosh\{(2n+1)\pi^2/(2\beta)\}} \\ &+ \pi \sum_{n=-\infty}^{\infty} \frac{(-1)^n \tanh\{(2n+1)\beta/2\}}{(2n+1)^2} - \frac{\beta\pi^2}{4} = 0. \end{aligned} \tag{10.2.19}$$

The second primary idea is to introduce a function f of a complex variable and use contour integration to prove (10.2.19). To that end, define, for fixed $\eta > 0$,

$$f(z) := \frac{\tan(\eta z)}{z^2 \cosh z}. \tag{10.2.20}$$

The function $f(z)$ is meromorphic in the entire complex plane with a simple pole at $z = 0$ and simple poles at $z = (2n+1)\pi i/2$ and $z = (2n+1)\pi/(2\eta)$ for each integer n. Let γ_{R_m} be a sequence of positively oriented circles centered at the origin and with radii R_m tending to ∞ as $m \to \infty$, where the radii R_m

are chosen so that the circles remain at a bounded distance from all the poles of $f(z)$. From the definition (10.2.20) of f, it is then easy to see that

$$\left| \int_{\gamma_{R_m}} f(z)dz \right| \ll_\eta \frac{1}{R_m}, \tag{10.2.21}$$

as $R_m \to \infty$, where the constant implied in the notation $\ll_\eta$ depends on η.

For brevity, let $R(a)$ denote the residue of $f(z)$ at a pole a. Then, brief calculations show that

$$R(0) = \eta, \tag{10.2.22}$$

$$R\left(\frac{(2n+1)\pi}{2\eta}\right) = -\frac{4\eta}{\pi^2(2n+1)^2\cosh\{(2n+1)\pi/(2\eta)\}}, \tag{10.2.23}$$

$$R\left(\frac{(2n+1)\pi i}{2}\right) = -\frac{4(-1)^n\tanh\{(2n+1)\pi\eta/2\}}{\pi^2(2n+1)^2}, \tag{10.2.24}$$

for each integer n. Hence, using (10.2.22)–(10.2.24) and the residue theorem, we deduce that

$$\begin{aligned}\frac{1}{2\pi i}\int_{\gamma_{R_m}} f(z)dz = \eta &- \sum_{|2n+1|<2\eta R_m/\pi} \frac{4\eta}{\pi^2(2n+1)^2\cosh\{(2n+1)\pi/(2\eta)\}} \\ &- \sum_{|2n+1|<2R_m/\pi} \frac{4(-1)^n\tanh\{(2n+1)\pi\eta/2\}}{\pi^2(2n+1)^2}.\end{aligned} \tag{10.2.25}$$

Letting R_m tend to ∞ in (10.2.25) and employing (10.2.21), we conclude that

$$\begin{aligned}0 = \eta &- \frac{4\eta}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{1}{(2n+1)^2\cosh\{(2n+1)\pi/(2\eta)\}} \\ &- \frac{4}{\pi^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n\tanh\{(2n+1)\pi\eta/2\}}{(2n+1)^2}.\end{aligned} \tag{10.2.26}$$

Now set $\eta = \beta/\pi$ in (10.2.26). Then multiply both sides by $-\pi^3/4$. We then readily obtain (10.2.19), and so this completes the proof. □

10.3 Further Remarks on Ramanujan's Quasi-theta Product

Theorem 10.2.1 can easily be generalized in at least two directions.

First, in the proof of (10.2.1), we could replace $f(z)$ by

$$f_n(z) := \frac{\tan(\eta z)}{z^n\cosh z},$$

where n is a positive integer exceeding 1. The generalization of (10.2.1) would then involve Bernoulli numbers arising from the Taylor expansion of $\tan z$ about $z=0$ and Euler numbers arising from the expansion of $1/\cosh z$ about $z=0$.

Second, in the proof of (10.2.1), we could replace $f(z)$ by

$$f(z,\theta) := \frac{\cosh(\theta z)\tan(\eta z)}{z^2\cosh z},$$

where $-1<\theta<1$. Then by a proof analogous to that given above, we could deduce that for any complex numbers α and β with $\operatorname{Re}\alpha, \operatorname{Re}\beta \neq 0$, and $\alpha\beta=\pi^2$, and for any real number θ with $|\theta|<1$,

$$\begin{aligned}
&\alpha\sum_{n=0}^{\infty}\frac{\sinh\{(2n+1)\alpha/2\}\cosh\{(2n+1)\theta\alpha/2\}}{(2n+1)\cosh^2\{(2n+1)\alpha/2\}}\\
&\quad-\theta\alpha\sum_{n=0}^{\infty}\frac{\sinh\{(2n+1)\theta\alpha/2\}}{(2n+1)\cosh\{(2n+1)\alpha/2\}}+\pi\sum_{n=0}^{\infty}\frac{(-1)^n\cos\{(2n+1)\pi\theta/2\}}{(2n+1)\cosh^2\{(2n+1)\beta/2\}}\\
&=\frac{\pi^2}{4}-2\sum_{n=0}^{\infty}\frac{\cosh\{(2n+1)\theta\alpha/2\}}{(2n+1)^2\cosh\{(2n+1)\alpha/2\}}.
\end{aligned}\tag{10.3.1}$$

The identity (10.3.1) is equivalent to

$$\begin{aligned}
&\left\{\prod_{n=0}^{\infty}\left(\frac{(1-(-1)^nq^{(2n+1-\theta)/2})(1-(-1)^nq^{(2n+1+\theta)/2})}{(1+(-1)^nq^{(2n+1-\theta)/2})(1+(-1)^nq^{(2n+1+\theta)/2})}\right)^{2n+1}\right\}^{(\log q)/2}\\
&\qquad\times\left\{\prod_{n=0}^{\infty}\frac{(1-(-1)^nq^{(2n+1-\theta)/2})(1+(-1)^nq^{(2n+1+\theta)/2})}{(1+(-1)^nq^{(2n+1-\theta)/2})(1-(-1)^nq^{(2n+1+\theta)/2})}\right\}^{-\theta(\log q)/2}\\
&\qquad\times\left\{\prod_{n=1}^{\infty}\left(\frac{(1+(-1)^nie^{\theta\pi i/2}q'^n)(1+(-1)^nie^{-\theta\pi i/2}q'^n)}{(1-(-1)^nie^{\theta\pi i/2}q'^n)(1-(-1)^nie^{-\theta\pi i/2}q'^n)}\right)^{n}\right\}^{\pi i}\\
&\qquad=\exp\left(\frac{\pi^2}{4}-2\sum_{n=0}^{\infty}\frac{\cosh\{(2n+1)\theta\alpha/2\}}{(2n+1)^2\cosh\{(2n+1)\alpha/2\}}\right).
\end{aligned}\tag{10.3.2}$$

When $\theta=0$, (10.3.1) and (10.3.2) reduce to (10.2.1) and (10.1.1), respectively. If $\alpha,\beta>0$ and $\theta=: u+iv$, where u and v are real, then (10.3.2) can be analytically continued to the rectangle $-1<u<1, -2\pi/\alpha<v<2\pi/\alpha$.

If we differentiate (10.3.1) $2k$ times with respect to θ and then set $\theta=0$, we deduce that

$$\alpha\sum_{n=0}^{\infty}\frac{(2n+1)^{2k-1}\sinh\{(2n+1)\alpha/2\}}{\cosh^2\{(2n+1)\alpha/2\}}\tag{10.3.3}$$

$$+ (-1)^k \beta^{2k} \pi^{1-2k} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)^{2k-1}}{\cosh^2\{(2n+1)\beta/2\}} = (4k-2) \sum_{n=0}^{\infty} \frac{(2n+1)^{2k-2}}{\cosh\{(2n+1)\alpha/2\}},$$

which is valid for any integer $k \geq 1$ and any complex numbers α and β with $\operatorname{Re} \alpha, \operatorname{Re} \beta \neq 0$, and $\alpha\beta = \pi^2$.

If we let $\alpha \to \infty$ (or $\beta \to 0$) in (10.2.1), we deduce Leibniz's well-known evaluation

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} = \frac{\pi}{4},$$

while if we let $\alpha \to 0$ (or $\beta \to \infty$) in (10.2.1), we deduce Euler's well-known evaluation

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

(We remark that care must be taken in taking certain limits inside summation signs above.)

Ramanujan examined several other infinite series of hyperbolic functions in his notebooks [243] and lost notebook [244]. We cite two examples giving evaluations of series involving $\cosh z$ that are very similar to those arising above.

First, in Entry 16(x) of Chapter 17 in his second notebook [243], [54, p. 134], Ramanujan asserted that

$$\sum_{n=0}^{\infty} \frac{(2n+1)^2}{\cosh\{(2n+1)\pi/2\}} = \frac{\pi^{3/2}}{2\sqrt{2}\Gamma^6(\frac{3}{4})}. \tag{10.3.4}$$

In fact, it is shown in [54, pp. 134–138] that one can also evaluate in closed form the more general sum

$$\sum_{n=0}^{\infty} \frac{(2n+1)^{2m}}{\cosh\{(2n+1)\alpha/2\}}, \tag{10.3.5}$$

where m is a positive integer. However, the evaluations are in terms of $z := {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; k^2)$. (See [54, p. 101, equation (6.3)] for the relation between α and k, where in [54], $y = \alpha$.) Note that the sums (10.3.5) appear on the right side of (10.3.3), and so these evaluations also automatically yield evaluations for the left side of (10.3.3).

Second, the evaluation

$$\sum_{n=0}^{\infty} \frac{(2n+1)^2}{\cosh^2\{(2n+1)\pi/2\}} = \frac{\pi^2}{12\Gamma^8(\frac{3}{4})}$$

arises in Ramanujan's formulas for the power series coefficients of the reciprocals, or, more generally, quotients, of certain Eisenstein series [62, Corollary 3.9].

Multivariable generalizations of the products in (10.1.1) have been studied by Berndt and Zaharescu [80] and S. Kongsiriwong [194].

10.4 A Generalization of the Dedekind Eta Function

On page 330 of [244], Ramanujan defines a generalization of the Dedekind eta function, states a transformation formula for it, and then gives three examples. The first of the three examples is the transformation formula for the Dedekind eta function. We first give Ramanujan's definition, his transformation formula, and his three examples. We then state and prove a theorem of E. Krätzel [195] that contains, as a special case, Ramanujan's transformation formula, but in a slightly different form.

Define, for each positive integer s and $x > 0$,

$$\phi_s(x) := \sqrt{2\pi x}\, e^{\pi x \zeta(-s)} \prod_{n=1}^{\infty} (1 - e^{-2\pi x n^s}),$$

where $\zeta(s)$ denotes the Riemann zeta function.

Entry 10.4.1 (p. 330). *For each positive integer s and $x > 0$,*

$$\begin{aligned}\phi_s(x) = (2\pi)^{s/2} \exp\left(\frac{\pi\zeta(-1/s)}{x^{1/s}\sin(\frac{\pi}{2s})}\right) \prod_{j=0}^{s-1}\prod_{n=1}^{\infty} \\ \left\{1 - 2\exp\left(-2\pi\left(\frac{n}{x}\right)^{1/s}\sin\left(\frac{\pi(2j+1)}{2s}\right)\right)\right. \\ \times\cos\left(2\pi\left(\frac{n}{x}\right)^{1/s}\cos\left(\frac{\pi(2j+1)}{2s}\right)\right) \\ \left.+\exp\left(-4\pi\left(\frac{n}{x}\right)^{1/s}\sin\left(\frac{\pi(2j+1)}{2s}\right)\right)\right\}^{1/2}.\end{aligned} \tag{10.4.1}$$

We now state the special cases $s = 1, 2, 3$ of Entry 10.4.1.

Entry 10.4.2 (p. 330).

$$\sqrt{x}e^{-\pi x/12}\prod_{n=1}^{\infty}(1 - e^{-2\pi n x}) = e^{-\pi/(12x)}\prod_{n=1}^{\infty}(1 - e^{-2\pi n/x}), \tag{10.4.2}$$

$$\begin{aligned}&\frac{1}{2}\sqrt{\frac{x}{\pi}}\exp\left(-\frac{2\pi\zeta(-1/2)}{\sqrt{x}}\right)\prod_{n=1}^{\infty}(1 - e^{-\pi x n^2}) \\ &\qquad = \prod_{n=1}^{\infty}\left\{1 - 2e^{-2\pi\sqrt{n/x}}\cos\left(2\pi\sqrt{n/x}\right) + e^{-4\pi\sqrt{n/x}}\right\},\end{aligned} \tag{10.4.3}$$

$$\begin{aligned}&\sqrt{x}\exp\left(\frac{\pi x}{120} - \frac{2\pi\zeta(-1/3)}{\sqrt[3]{x}}\right)\prod_{n=1}^{\infty}(1 - e^{-2\pi n^3 x}) \\ &= 2\pi\prod_{n=1}^{\infty}\left\{\left(1 - e^{-2\pi\sqrt[3]{n/x}}\right)\left(1 - 2e^{-\pi\sqrt[3]{n/x}}\cos\left(\pi\sqrt{3}\sqrt[3]{n/x}\right) + e^{-2\pi\sqrt[3]{n/x}}\right)\right\}.\end{aligned} \tag{10.4.4}$$

The transformation formula (10.4.2) follows immediately from (10.4.1) by setting $s = 1$, and is the familiar transformation formula for the Dedekind eta function [53, p. 256, Corollary (ii)], [54, p. 43, Entry 27(iii)]. The formula (10.4.3) follows readily from (10.4.1) when x is replaced by $x/2$. We have corrected a minor misprint in the first factor of the infinite product on the right side of (10.4.4), which arises from (10.4.1) with $s = 3$. In the calculation of (10.4.4), we need the value $\zeta(-3) = \frac{1}{120}$ [271, p. 19]. We also note that the terms with $j = 0$ and $j = 2$ are identical and that the term with $j = 1$ is a perfect square when $s = 3$.

A result equivalent to (10.4.3) was given without proof by G.H. Hardy and Ramanujan in Section 7.3 of [176]. It was rediscovered by D. Kim [190] and was also proved by R. Baxter [49], who proved several results of this nature. Finally, E.M. Wright utilized a transformation formula equivalent to Entry 10.4.1 for his treatment of partitions into powers [284].

We now state and prove Krätzel's theorem, generalizing Entry 10.4.1. For relatively prime positive integers a and b and $|\arg t| < \pi/(2ab)$, define the generalized Dedekind eta function

$$\eta_{a,b}(t) := (2\pi)^{(1-b)/2} e^{\gamma_{a,b}(t)} \prod_{n=1}^{\infty} \prod_{\nu=0}^{a-1} (1 - e^{2\pi i \epsilon_{2\nu+1}(4a) n^{b/a} t^b}), \tag{10.4.5}$$

where

$$\epsilon_\nu(n) := e^{2\pi i \nu / n} \tag{10.4.6}$$

and

$$\gamma_{a,b}(t) := \frac{\pi \zeta(-b/a)}{\sin(\pi/(2a))} t^b, \tag{10.4.7}$$

where $\zeta(s)$ again denotes the Riemann zeta function. Observe that $\eta_{1,1} = \eta(t)$, the ordinary Dedekind eta function.

Theorem 10.4.1. *If a and b are positive integers with $(a, b) = 1$ and $|\arg t| < \pi/(2ab)$, then*

$$\eta_{a,b}(t) = t^{-ab/2} \eta_{b,a}(1/t). \tag{10.4.8}$$

Proof. Taking the logarithm of both sides of (10.4.5), we find that

$$\log \eta_{a,b}(t) = \frac{1-b}{2} \log(2\pi) + \gamma_{a,b}(t) - \sum_{\nu=0}^{a-1} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m} e^{2\pi i \epsilon_{2\nu+1}(4a) n^{b/a} m t^b}. \tag{10.4.9}$$

Recall the integral representation

$$e^{-t} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s) t^{-s} ds, \tag{10.4.10}$$

where $\operatorname{Re} t > 0$ and $c > 0$. Using (10.4.10) in (10.4.9) with t replaced by $2\pi\epsilon_{2\nu+1-a}(4a)t^b$ and with $c > a/b$, noting that the hypothesis $|\arg t| <$

$\pi/(2ab)$ ensures the applicability of (10.4.10), inverting the order of summation and integration, and twice using the familiar series representation of the Riemann zeta function, we find that

$$\begin{aligned}\log \eta_{a,b}(t) &= \frac{1-b}{2}\log(2\pi) + \gamma_{a,b}(t)\\ &\quad - \sum_{\nu=0}^{a-1} \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s+1)\zeta(bs/a)\left(2\pi\epsilon_{2\nu+1-a}(4a)t^b\right)^{-s} ds\\ &= \frac{1-b}{2}\log(2\pi) + \gamma_{a,b}(t)\\ &\quad - \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s+1)\zeta(bs/a)\frac{\sin(\pi s/2)}{\sin(\pi s/(2a))}(2\pi t^b)^{-s} ds,\end{aligned} \tag{10.4.11}$$

where in the last step we inverted the order of summation to sum the a roots of unity. Recall the functional equation of the Riemann zeta function given by [271, p. 25]

$$\zeta(s+1) = -(2\pi)^s \frac{\pi\zeta(-s)}{\Gamma(s+1)\sin(\pi s/2)}. \tag{10.4.12}$$

Using (10.4.12) in (10.4.11) along with the functional equation $\Gamma(z+1) = z\Gamma(z)$, we find that

$$\log \eta_{a,b}(t) = \frac{1-b}{2}\log(2\pi) + \gamma_{a,b}(t) + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\pi\zeta(-s)\zeta(bs/a)}{s\sin(\pi s/(2a))} t^{-bs} ds.$$

Replacing s by as above, we arrive at

$$\begin{aligned}\log \eta_{a,b}(t) &= \frac{1-b}{2}\log(2\pi) + \gamma_{a,b}(t)\\ &\quad + \frac{1}{2\pi i}\int_{ca-i\infty}^{ca+i\infty} \frac{\pi\zeta(-as)\zeta(bs)}{s\sin(\pi s/2)} t^{-abs} ds.\end{aligned} \tag{10.4.13}$$

We now shift the line of integration to $-c_0 - i\infty, -c_0 + i\infty$, $c_0 > 1/a$, by integrating over a rectangle C_T with vertices $c \pm iT, -c_0 \pm iT$, where $T > 0$, applying the residue theorem, and letting $T \to \infty$. Because, uniformly in any vertical strip [271, p. 81], $\zeta(\sigma + iT) = O(T^k)$, as $T \to \infty$, for some constant k that may depend on the particular vertical strip, we easily see that the integrals over the horizontal sides of C_T tend to 0 as T tends to ∞. On the interior of C_T the integrand on the right side of (10.4.13) has simple poles at $s = 1/b$ and $-1/a$ and a double pole at $s = 0$. Let R_α denote the residue of a pole at α. Lastly, replace s by $-s$. After all this, we deduce from (10.4.13) that

$$\log \eta_{a,b}(t) = \frac{1-b}{2}\log(2\pi) + \gamma_{a,b}(t) + R_{1/b} + R_0 + R_{-1/a}$$

$$+\frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty}\frac{\pi\zeta(as)\zeta(-bs)}{s\sin(\pi s/2)}t^{abs}ds. \tag{10.4.14}$$

Straightforward calculations show that

$$R_{-1/a}=-\gamma_{a,b}(t) \qquad \text{and} \qquad R_{1/b}=\gamma_{b,a}(1/t). \tag{10.4.15}$$

Since [271, p. 20]

$$\zeta(s)=-\frac{1}{2}-\frac{1}{2}\log(2\pi)s+\cdots,$$

we find that

$$R_0=\frac{1}{2}(b-a)\log(2\pi)-\frac{1}{2}ab\log t. \tag{10.4.16}$$

Putting (10.4.15) and (10.4.16) in (10.4.14), we deduce that

$$\begin{aligned}\log\eta_{a,b}(t)=&\frac{1-a}{2}\log(2\pi)+\gamma_{b,a}(1/t)-\frac{1}{2}ab\log t\\&+\frac{1}{2\pi i}\int_{c_0-i\infty}^{c_0+i\infty}\frac{\pi\zeta(as)\zeta(-bs)}{s\sin(\pi s/2)}t^{abs}ds.\end{aligned} \tag{10.4.17}$$

Comparing (10.4.17) with (10.4.13), we see that on the right side of (10.4.17) the roles of a and b have been reversed, there is an extra additive factor of $-\frac{1}{2}ab\log t$, and that t has been replaced by $1/t$. In other words, upon exponentiation, we obtain (10.4.8). □

We now show that Krätzel's theorem yields Entry 10.4.1. First, observe that in (10.4.1), the terms with $j=\nu$ and $j=s-\nu-1$ are identical. Thus, if s is even, the first product in (10.4.1) can be taken over $0\le j\le s/2-1$, and the square root on the product on the right side can be deleted. If s is odd, then the term with $j=(s-1)/2$ is, in fact, a perfect square.

Now examine Theorem 10.4.1, with t^b replaced by x, $a=1$, and $b=s$. Consider the product of the terms with index ν and $s-\nu-1$ on the right side. These are

$$\begin{aligned}&(1-e^{2\pi i\left(\cos\frac{\pi\nu}{2s}+i\sin\frac{\pi\nu}{2s}\right)(n/x)^{1/s}})(1-e^{2\pi i\left(-\cos\frac{\pi\nu}{2s}+i\sin\frac{\pi\nu}{2s}\right)(n/x)^{1/s}})\\&\qquad=1-2e^{-2\pi(n/x)^{1/s}\sin\frac{\pi\nu}{2s}}\cos(2\pi(n/x)^{1/s}\cos\tfrac{\pi\nu}{2s})+e^{-4\pi(n/x)^{1/s}\sin\frac{\pi\nu}{2s}}.\end{aligned}$$

The term with $\nu=(s-1)/2$ in Theorem 10.4.1 is identical to the corresponding term in (10.4.1). Thus, when simplified, Ramanujan's Entry 10.4.1 becomes identical with Theorem 10.4.1 when simplified, after the parameters are specialized and changed as described above. With these observations, we restate Entry 10.4.1.

Entry 10.4.3 (p. 330). *For each positive even integer s and $x>0$,*

$$\phi_s(x)=(2\pi)^{s/2}\exp\left(\frac{\pi\zeta(-1/s)}{x^{1/s}\sin(\frac{\pi}{2s})}\right)\prod_{j=0}^{s/2-1}\prod_{n=1}^{\infty}$$

$$\left\{1 - 2\exp\left(-2\pi\left(\frac{n}{x}\right)^{1/s}\sin\left(\frac{\pi(2j+1)}{2s}\right)\right)\right.$$
$$\times\cos\left(2\pi\left(\frac{n}{x}\right)^{1/s}\cos\left(\frac{\pi(2j+1)}{2s}\right)\right)$$
$$\left.+\exp\left(-4\pi\left(\frac{n}{x}\right)^{1/s}\sin\left(\frac{\pi(2j+1)}{2s}\right)\right)\right\}^{1/2};$$

For each positive odd integer s and $x > 0$,

$$\phi_s(x) = (2\pi)^{s/2}\exp\left(\frac{\pi\zeta(-1/s)}{x^{1/s}\sin(\frac{\pi}{2s})}\right)(1 - e^{-2\pi(n/x)^{1/s}})\prod_{j=0}^{(s-3)/2}\prod_{n=1}^{\infty}$$
$$\left\{1 - 2\exp\left(-2\pi\left(\frac{n}{x}\right)^{1/s}\sin\left(\frac{\pi(2j+1)}{2s}\right)\right)\right.$$
$$\times\cos\left(2\pi\left(\frac{n}{x}\right)^{1/s}\cos\left(\frac{\pi(2j+1)}{2s}\right)\right)$$
$$\left.+\exp\left(-4\pi\left(\frac{n}{x}\right)^{1/s}\sin\left(\frac{\pi(2j+1)}{2s}\right)\right)\right\}^{1/2}.$$

10.5 Two Entries on Page 346

We recall Ramanujan's notation in the theory of elliptic functions [54, p. 101]. Let

$$z := {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1, k^2), \tag{10.5.1}$$

where ${}_2F_1$ denotes the ordinary or Gaussian hypergeometric function, and k, $0 < k < 1$, denotes the *modulus*. Furthermore, put

$$q := e^{-y}, \tag{10.5.2}$$

where

$$y := \pi\frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1, 1-k^2)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1, k^2)}. \tag{10.5.3}$$

Recall that the *complementary modulus* k' is defined by $k' = \sqrt{1-k^2}$, $0 < k' < 1$.

Entry 10.5.1 (p. 346). *For q defined by* (10.5.2) *and θ and ϕ defined by*

$$z\theta = \int_0^{\phi}\frac{du}{\sqrt{1-k^2\sin^2 u}}, \tag{10.5.4}$$

we have

$$\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + 4\sum_{n=0}^{\infty}\frac{(-1)^n q^{2n+1}\sin(2n+1)\theta}{(2n+1)(1-q^{2n+1})} = \log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right).$$

Entry 10.5.1 is identical to Entry 16(v) in Chapter 18 of Ramanujan's second notebook [243], [54, p. 175].

For reasons that will become apparent in the next proof, we replace Ramanujan's θ' in the next entry by θ^*.

Entry 10.5.2 (p. 346). *Define* θ^* *and* ϕ *by*

$$z\theta^* := \int_0^{\phi} \frac{du}{\sqrt{1-k'^2\sin^2 u}}, \tag{10.5.5}$$

where z *and* k' *are as above. Then*

$$\theta^* + 2\sum_{n=1}^{\infty} \frac{q^n \sinh(2n\theta^*)}{n(1+q^{2n})} = \log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right). \tag{10.5.6}$$

Proof. We begin with the following principle in the theory of elliptic functions found in Section 18 of Chapter 18 in Ramanujan's second notebook [243], [54, pp. 177–179]. Suppose that we have an equation of the sort

$$\Omega(k, e^{-y}, z, \theta, \phi) = 0. \tag{10.5.7}$$

We now want to write a new equation with k replaced by k'. From (10.5.1), we see that z will be replaced by

$$z' := {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1, k'^2); \tag{10.5.8}$$

from (10.5.3), y will be replaced by

$$y' := \pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1, 1-k'^2)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1, k'^2)}; \tag{10.5.9}$$

and from Entry 18(iv) of Chapter 18 in the second notebook [243], [54, p. 179], θ will be replaced by

$$\theta' := i\theta z/z', \tag{10.5.10}$$

and ϕ will be replaced by

$$\phi' := i\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right). \tag{10.5.11}$$

Thus, we obtain a new equation

$$\Omega\left(k', e^{-y'}, z', i\theta z/z', i\log\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right) = 0. \tag{10.5.12}$$

We are going to apply this principle, but with the roles of the variables reversed. Thus, taking Entry 15(iv) of Chapter 18 in Ramanujan's second notebook [243], [54, p. 172], we replace the variables by their counterparts with primes $'$ on them. Thus, we have

$$\theta' + \sum_{n=1}^{\infty} \frac{\sin(2n\theta')}{n\cosh(ny')} = \phi'. \tag{10.5.13}$$

We now use (10.5.12) to convert (10.5.13) to an identity involving θ. Hence,

$$\frac{i\theta' z'}{z} + \sum_{n=1}^{\infty} \frac{\sin(2ni\theta' z'/z)}{n\cosh(ny)} = \phi' = i\log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right), \tag{10.5.14}$$

by (10.5.11). By (10.5.4), with the roles of the pairs θ, k and θ', k' reversed,

$$z'\theta' = \int_0^{\phi} \frac{du}{\sqrt{1 - k'^2 \sin^2 u}}. \tag{10.5.15}$$

Comparing (10.5.15) with (10.5.5), we see that $z'\theta' = z\theta^*$. Using this equality and (10.5.2) in (10.5.14), we deduce that

$$i\theta^* + 2i\sum_{n=1}^{\infty} \frac{q^n \sin(2n\theta^*)}{n(1+q^{2n})} = i\log\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right),$$

which upon canceling i throughout yields (10.5.6). $\square$

10.6 A Continued Fraction

Entry 10.6.1 (p. 370). *Let $K = K(k)$ denote the complete elliptic integral of the first kind associated with the modulus k, and let $K' = K(k')$ be the complete elliptic integral of the first kind associated with the complementary modulus $k' = \sqrt{1-k^2}$. Then, if $n > 0$,*

$$\frac{\pi n K}{2}\left\{\frac{1}{n^2K^2} + 4\sum_{j=1}^{\infty}\left(\frac{q^j}{1+q^{2j}} \cdot \frac{1}{n^2K^2 + \pi^2 j^2}\right)\right\} = \frac{1}{n+}\,\frac{k^2}{n+}\,\frac{2^2}{n+}\,\frac{(3k)^2}{n+}\,\frac{4^2}{n+}\,\frac{(5k)^2}{n+\cdots}. \tag{10.6.1}$$

Proof. In Entry 12(i) of Chapter 18 in his second notebook, Ramanujan recorded the continued fraction

$$\frac{1}{2} + \sum_{j=1}^{\infty} \frac{\operatorname{sech}(jy)}{1+(jm)^2} = \frac{z}{2+}\,\frac{(mz)^2 x}{2+}\,\frac{(2mz)^2}{2+}\,\frac{(3mz)^2 x}{2+}\,\frac{(4mz)^2}{2+\cdots}, \tag{10.6.2}$$

where $z := 2K/\pi$, $x = k^2$, $y = \pi K'/K$, and $m > 0$. For a proof, see [54, p. 163]. Multiply both sides of (10.6.2) by 2 and replace mz by $2/n$ to obtain the equivalent continued fraction

$$1+2\sum_{j=1}^{\infty}\frac{\operatorname{sech}(jy)}{1+(2j/(nz))^2}=\frac{z}{1\,+}\,\frac{(k/n)^2}{1}\,\frac{}{+}\,\frac{(2/n)^2}{1}\,\frac{}{+}\,\frac{(3k/n)^2}{1}\,\frac{}{+}\,\frac{(4/n)^2}{1}\,\frac{}{+\cdots}. \tag{10.6.3}$$

Multiplying both sides of (10.6.3) by $1/n$, using the definition $z := 2K/\pi$, and rearranging, we find the equivalent continued fraction

$$\frac{1}{n}\left(1+\frac{2n^2K^2}{\pi^2}\sum_{j=1}^{\infty}\frac{\operatorname{sech}(jy)}{(nK/\pi)^2+j^2}\right)$$
$$=\frac{2K/\pi}{n}\,\frac{}{+}\,\frac{k^2}{n}\,\frac{}{+}\,\frac{2^2}{n}\,\frac{}{+}\,\frac{(3k)^2}{n}\,\frac{}{+}\,\frac{4^2}{n}\,\frac{}{+\cdots}. \tag{10.6.4}$$

Lastly, multiply both sides of (10.6.4) by $\pi/(2K)$, set $q = e^{-y}$ in the definition of sech, and rearrange slightly to achieve (10.6.1). □

The previous entry has a fascinating corollary found in Section 12 of Chapter 18 of Ramanujan's second notebook [243], [54, p. 164].

10.7 Class Invariants

We begin by recalling the definitions of class invariants by both Ramanujan and H. Weber [281]. As usual, set

$$\chi(q) := (-q; q^2)_\infty.$$

For

$$q := \exp(-\pi\sqrt{n}),$$

where n is a natural number, Ramanujan's class invariants G_n and g_n are defined by

$$G_n := 2^{-1/4}q^{-1/24}\chi(q) \qquad \text{and} \qquad g_n := 2^{-1/4}q^{-1/24}\chi(-q).$$

In the notation of Weber,

$$G_n = 2^{-1/4}\mathfrak{f}(\sqrt{-n}) \qquad \text{and} \qquad g_n = 2^{-1/4}\mathfrak{f}_1(\sqrt{-n}).$$

Ramanujan devoted considerable energy to calculating over 100 class invariants. An account of most of Ramanujan's work can be found in Chapter 34 of [57].

On page 342 in [244], Ramanujan provides a list of class invariants from Weber's book [281, pp. 722–724]. It is clear that he did not merely copy this list of invariants, because at the top of page 342, Ramanujan lists two mistakes that Weber made in calculating $\mathfrak{f}(\sqrt{-41})$. Weber's table [281, pp. 721–726] contains further incorrect values of class invariants. J. Brillhart and P. Morton carefully checked the entire table and published a complete list of errors [95], which, of course, contains the mistakes noticed much earlier by Ramanujan. Ramanujan's methods for calculating class invariants have largely remained in impenetrable darkness, and so it is again unfortunate that Ramanujan left no clues about his methods of calculation. We now provide this table below, with two trivial misprints corrected.

Entry 10.7.1 (p. 342).

$$\begin{aligned}
\mathfrak{f}_1(\sqrt{-20}) &= t\sqrt[8]{8}, &\quad& t^8 - \frac{1+\sqrt{5}}{2}(2t^4+1) = 0.\\
\mathfrak{f}(\sqrt{-27}) &= t\sqrt[3]{2}, && t^9 - 3t^6 - 3t^3 - 1 = 0.\\
\mathfrak{f}_1(\sqrt{-26}) &= t\sqrt[4]{2}, && t^6 - t^4 - \frac{3+\sqrt{13}}{2}(t^2+1) = 0.\\
\mathfrak{f}(\sqrt{-29}) &= t\sqrt[4]{2}, && 2t^{12} - 9t^8 - 8t^4 - 5 = \sqrt{29}(t^4+1)^2.\\
\mathfrak{f}(\sqrt{-35}) &= t, && t^3 - 2 = (1+\sqrt{5})(t^2 - t).\\
\mathfrak{f}_1(\sqrt{-36}) &= t\sqrt[8]{2}, && t^6 - 4t^3 - 2 = 2\sqrt{3}(t^3+1).\\
\mathfrak{f}_1(\sqrt{-38}) &= t\sqrt[4]{2}, && t^6 - 2t^4 - (2t^2+1)(1+\sqrt{2}) = 0.\\
\mathfrak{f}(\sqrt{-39}) &= t\sqrt{2}, && t^6 - \frac{3+\sqrt{13}}{2}(t^3+1) = 0.\\
\mathfrak{f}(\sqrt{-41}) &= t\sqrt[4]{2}, && \left(t^2+\frac{1}{t^2}\right)^2 - \frac{5+\sqrt{41}}{2}\left(t^2+\frac{1}{t^2}\right) + \frac{7+\sqrt{41}}{2} = 0.\\
\mathfrak{f}_1(\sqrt{-44}) &= t\sqrt[8]{8}, && t^{12} - (6+2\sqrt{11})t^8 + (8+2\sqrt{11})t^4 - (3+\sqrt{11}) = 0.\\
\mathfrak{f}_1(\sqrt{-50}) &= t\sqrt[4]{2}, && t^3 - t^2 = \frac{1+\sqrt{5}}{2}(t+1).\\
\mathfrak{f}(\sqrt{-51}) &= t\sqrt[3]{2}, && t^9 - (4+\sqrt{17})t^6 - t^3 - 1 = 0.\\
\mathfrak{f}_1(\sqrt{-52}) &= t\sqrt[8]{8}, && t^8 - 2(4+\sqrt{13})t^4 - \frac{3+\sqrt{13}}{2} = 0.\\
\mathfrak{f}(\sqrt{-63}) &= t\sqrt{2}, && \frac{t^6+3t^3-1}{t^6-t^3+1} = \sqrt{\frac{7}{3}}.\\
\mathfrak{f}(\sqrt{-99}) &= t\sqrt[3]{2}, && t^9 - (12+2\sqrt{33})t^6 - (4+\sqrt{33})t^3 - 1 = 0.
\end{aligned}$$

11

Formulas for the Power Series Coefficients of Certain Quotients of Eisenstein Series

11.1 Introduction

In their epic paper [176], [242, pp. 276–309], G.H. Hardy and S. Ramanujan found an asymptotic formula for the partition function $p(n)$ that arises from the power series coefficients of the reciprocal of the Dedekind eta function. As they indicated near the end of their paper, their methods also apply to several analogues of the partition function generated by modular forms of negative weight that are analytic in the upper half-plane. In their last published paper [177], [242, pp. 310–321], they considered a similar problem for the coefficients of modular forms of negative weight having a simple pole in a fundamental region, and in particular, they applied their theorem to find interesting series representations for the coefficients of the reciprocal of the Eisenstein series $E_6(\tau)$. Although there are some similarities in the methods of these papers, the principal ideas are quite different in [177] from those in [176]. In [176], Hardy and Ramanujan introduced their famous circle method, and since that time the ideas in this paper have had an enormous impact in additive analytic number theory. Although their paper [177] has not had as much influence, the ideas in [177] have been extended by, among others, J. Lehner [201], H. Petersson [228], [229], [230], H. Poincaré [231], and H.S. Zuckerman [291]. Additional comments on [177] can be found in the third edition of [242, p. 387].

While confined to nursing homes and sanitariums during his last two years in England, Ramanujan wrote several letters to Hardy about the coefficients in the power series expansions of certain quotients of Eisenstein series. A few pages in his lost notebook are also devoted to this topic. All of this material can be found in [244, pp. 97–126], and the letters with commentary can be found in the book by Berndt and R.A. Rankin [74, pp. 175–191]. In these letters and in the lost notebook, Ramanujan claims formulas for the coefficients of several quotients of Eisenstein series not examined by Hardy and him in [177]. In fact, for some of these quotients, the main theorem of [177] needs to be moderately modified and improved. For other examples, a significantly stronger theorem is necessary. Ramanujan obviously wanted another exam-

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ple to be included in their paper [177], for in his letter of 28 June 1918 [74, pp. 182–183], he wrote, "I am sending you the analogous results in case of g_2. Please mention them in the paper without proof. After all we have got only two neat examples to offer, viz. g_2 and g_3. So please don't omit the results." This letter was evidently written after galley proofs for [177] were printed, because Ramanujan's request went unheeded. The functions g_2 and g_3, defined in (9.2.2) and (9.2.3), respectively, are the familiar invariants in the theory of elliptic functions and are constant multiples of the Eisenstein series $E_4(\tau)$ and $E_6(\tau)$, respectively. This letter was also evidently written before Ramanujan obtained further examples.

In this chapter, we establish the formulas for the coefficients of those quotients of Eisenstein series found in [244, pp. 102–104, 117]. In Ramanujan's notation, the three relevant Eisenstein series are defined, for $|q| < 1$, by

$$P(q) := 1 - 24\sum_{k=1}^{\infty} \frac{kq^k}{1-q^k}, \tag{11.1.1}$$

$$Q(q) := 1 + 240\sum_{k=1}^{\infty} \frac{k^3q^k}{1-q^k}, \tag{11.1.2}$$

and

$$R(q) := 1 - 504\sum_{k=1}^{\infty} \frac{k^5q^k}{1-q^k}. \tag{11.1.3}$$

(The notation above is that used in Ramanujan's paper [240], [242, pp. 136–162] and in his lost notebook [244]. In his notebooks [243], Ramanujan replaced P, Q, and R by L, M, and N, respectively.) In more contemporary notation, the Eisenstein series $E_{2j}(\tau)$ is defined for $j > 1$ and $\operatorname{Im} \tau > 0$ by

$$\begin{aligned} E_{2j}(\tau) := \frac{1}{2} \sum_{\substack{m_1,m_2\in\mathbb{Z} \\ (m_1,m_2)=1}} (m_1\tau + m_2)^{-2j} &= 1 - \frac{4j}{B_{2j}} \sum_{k=1}^{\infty} \frac{k^{2j-1}q^k}{1-q^k} \\ &= 1 - \frac{4j}{B_{2j}} \sum_{r=1}^{\infty} \sigma_{2j-1}(r)q^r, \end{aligned} \tag{11.1.4}$$

where $q = e^{2\pi i\tau}$, $B_j, j \geq 0$, denotes the jth Bernoulli number, and $\sigma_\nu(n) = \sum_{d|n} d^\nu$. Thus, for $q = \exp(2\pi i\tau)$, $E_4(\tau) = Q(q)$ and $E_6(\tau) = R(q)$, which have weights 4 and 6, respectively [255, p. 50]. Since (11.1.4) does not converge for $j = 1$, the Eisenstein series $E_2(\tau)$ must be defined differently. First let

$$E_2^*(\tau) := P(q), \qquad q = e^{2\pi i\tau}. \tag{11.1.5}$$

Then $E_2(\tau)$ is defined by

$$E_2(\tau) := E_2^*(\tau) - \frac{3}{\pi \operatorname{Im} \tau}. \tag{11.1.6}$$

The function $E_2(\tau)$ satisfies the functional equation of a modular form of weight 2 [255, pp. 67–68], but it is not a modular form.

Central for our proofs are Ramanujan's differential equations for Eisenstein series, namely [240, equations (30)], [242, p. 142],

$$q\frac{dP}{dq} = \frac{P^2(q) - Q(q)}{12}, \tag{11.1.7}$$

$$q\frac{dQ}{dq} = \frac{P(q)Q(q) - R(q)}{3}, \tag{11.1.8}$$

and

$$q\frac{dR}{dq} = \frac{P(q)R(q) - Q^2(q)}{2}. \tag{11.1.9}$$

Next define

$$B(q) := 1 + 24\sum_{k=1}^{\infty} \frac{(2k-1)q^{2k-1}}{1 - q^{2k-1}}, \qquad |q| < 1. \tag{11.1.10}$$

As will be seen, $B(q)$ is the (unique) modular form of weight 2 with multiplier system identically equal to 1 on the modular group $\Gamma_0(2)$.

As indicated above, in [177], Hardy and Ramanujan obtained representations for the coefficients of $1/R(q)$ as infinite series. In this chapter, we first establish Ramanujan's similar claims for the series

$$\frac{1}{Q(q)}, \qquad \frac{Q(q)}{R(q)}, \qquad \frac{P(q)}{R(q)}, \qquad \frac{P^2(q)}{R(q)}, \qquad \text{and} \qquad \frac{P(q)}{Q(q)}.$$

To prove the results in this first class of formulas, a moderate modification of the theorem of Hardy and Ramanujan is needed. This amended theorem is proved in detail in Section 11.2. The next four sections are devoted to proofs of several of Ramanujan's formulas for coefficients that can be derived from this key theorem. Most of the results in the first part of this chapter, i.e., through Section 11.6, can be found in P. Bialek's doctoral dissertation [84], or in a paper by Berndt and Bialek [61].

The representations in the second class, namely, Ramanujan's formulas for the coefficients of $1/B(q)$ and $1/B^2(q)$, are much harder to prove. To establish the first main result, we need an extension of Hardy and Ramanujan's theorem due to Petersson [228]. To prove the second primary result, we need to first extend work of Poincaré [176], Petersson [228], [229], [230], and Lehner [201] to functions with double poles, which are not examined in the work of any of these authors. The contents of the second part of this chapter, wherein these two remarkable formulas are proved, i.e., beginning with Section 11.7, are taken from a paper by Berndt, Bialek, and A.J. Yee [62]. Three further formulas of Ramanujan follow from one of the two key formulas, and these are also proved here.

Ramanujan claims that the second set of assertions follow in part from eight identities for Eisenstein series and theta functions which he states without proofs at the beginning of his letter [74, pp. 189–190]. Indeed, these eight identities are used in our proofs.

In Section 11.7, we prove the eight identities cited above. Section 11.8 contains a proof of Ramanujan's formula for the coefficients of $1/B(q)$. In Section 11.9, we show that three of Ramanujan's claims are consequences of the claim proved in Section 11.8. Lastly, in Section 11.10, we first prove an analogue for double poles of Hardy and Ramanujan's chief theorem, after which we prove Ramanujan's formula for the coefficients of $1/B^2(q)$.

As we shall see in the sequel, the series found by Hardy and Ramanujan are very rapidly convergent, even more so than those arising from modular forms analytic in the upper half-plane, so that truncating a series, even with a small number of terms, provides a remarkable approximation. Using *Mathematica*, we calculated several coefficients and series approximations for the functions $1/B(q)$ and $1/B^2(q)$. As will be seen from the first table, the coefficient of q^{10} in $1/B(q)$, for example, has 17 digits, while just two terms of Ramanujan's infinite series representation calculate this coefficient with an error of approximately 0.0003. Although we do not provide details, we calculated the coefficients of $1/B^2(q)$ up to $n = 50$. To demonstrate the rapid convergence of Ramanujan's series, we remark that for $n = 20, 30, 40$, and 50, the coefficients have, respectively, 29, 43, 57, and 70 digits, while two-term approximations give, respectively, 29, 42, 55, and 66 of these digits.

One of Ramanujan's letters to Hardy [244, pp. 97–101] is devoted to establishing upper and lower bounds for the number of terms in the representation for the coefficients of $1/R(q)$ needed to explicitly determine the actual (integral) coefficients. Although details are given in [244], expanded arguments can be found in Bialek's thesis [84] and in Chapter 12 of this book.

Different kinds of formulas for coefficients of modular forms have recently been established by J.H. Bruinier, W. Kohnen, and K. Ono [96].

We complete the introduction by setting notation. The set of rational integers is denoted by $\mathbb{Z}$, with $\mathbb{Z}^+$ denoting the set of positive integers. The upper half-plane $\mathcal{H}$ is defined by

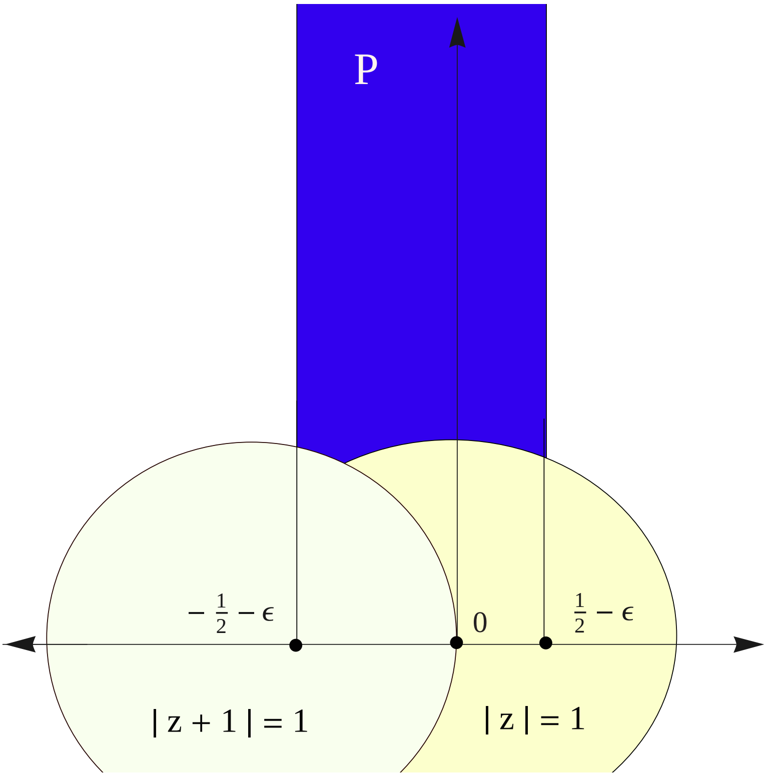
$$\mathcal{H} = \{\tau : \operatorname{Im} \tau > 0\}.$$

Throughout this chapter, we consider quotients of Eisenstein series that are not analytic in the upper half-plane. Each quotient is analytic in some disk, $|q| < q_0 < 1$, where q_0 is not necessarily the same at each appearance. The residue of a function $f(\tau)$ at a pole α is denoted by $\operatorname{Res}(f, \alpha)$. The full modular group is denoted by $\Gamma(1)$, and the modular subgroup $\Gamma_0(2)$ of $\Gamma(1)$ is defined by

$$\Gamma_0(2) = \left\{ T(\tau) = \frac{a\tau + b}{c\tau + d} \;\middle|\; a, b, c, d \in \mathbb{Z}; ad - bc = 1; c \text{ even} \right\}.$$

We let P_1 denote the fundamental region

$$P_1 = \{\tau : \text{Im } \tau > 0, \quad -\tfrac{1}{2} < \text{Re } \tau < \tfrac{1}{2}, \quad |\tau| > 1\}.$$



Further fundamental regions are P_2, the region in $\mathcal{H}$ bounded by the three circles $|\tau| = 1$, $|\tau + 1| = 1$, and $|\tau - 1| = 1$; P_3, the region in $\mathcal{H}$ bounded by the circles $|\tau - 1| = 1$ and $|\tau - \frac{1}{3}| = \frac{1}{3}$ and the line Re $\tau = \frac{1}{2}$; and P_4, the region in $\mathcal{H}$ bounded by the circle $|\tau + 1| = 1$ and the lines Re $\tau = -\frac{3}{2}$ and Re $\tau = -\frac{1}{2}$. However, the fundamental region most important for us is the fundamental region P in $\mathcal{H}$ bounded by the circles $|\tau + 1| = 1$ and $|\tau| = 1$ and the lines Re $\tau = -\frac{1}{2} - \epsilon$ and Re $\tau = \frac{1}{2} - \epsilon$, where $0 < \epsilon < 1$.

11.2 The Key Theorem

The principal tool in proving Ramanujan's formulas is the following theorem, which is essentially due to Hardy and Ramanujan [177], [242, pp. 312, 316].

However, we need to modify Hardy and Ramanujan's theorem slightly (by refining the estimate of their integral). In particular, in two of our applications, we examine $1/Q(q)$, which has a pole at a point on the boundary of P_1, and so we need to work on the fundamental region P instead of P_1, both defined at the end of Section 11.1. For the convenience of readers who may be reading this chapter while consulting or comparing it with Hardy and Ramanujan's paper [177], we have adhered to the notation of [177]. In particular, they set $q = e^{\pi i\tau}$ instead of the more customary $q = e^{2\pi i\tau}$ and therefore consider functions with arguments q^2.

Theorem 11.2.1. *Suppose that $f(q) = f(e^{\pi i\tau}) = \varphi(\tau)$ is analytic for $q = 0$, is meromorphic in the unit circle, and satisfies the functional equation*

$$\varphi(\tau) = \varphi\left(\frac{a\tau + b}{c\tau + d}\right)(c\tau + d)^n, \tag{11.2.1}$$

where a, b, c, $d \in \mathbb{Z}$; $ad - bc = 1$; and $n \in \mathbb{Z}^+$. If $\varphi(\tau)$ has only one pole in the fundamental region P_1, a simple pole at $\tau = \alpha$ with residue A, then

$$f(q) = -2\pi i A \sum \frac{1}{(c\alpha + d)^{n+2}} \frac{1}{1 - (q/\underline{q})^2}, \qquad |q| < q_0, \tag{11.2.2}$$

and

$$0 = -2\pi i A \sum \frac{1}{(c\alpha + d)^{n+2}} \frac{1}{1 - (q/\underline{q})^2}, \qquad |q| > 1, \tag{11.2.3}$$

where

$$\underline{q} = \exp\left(\left(\frac{a\alpha + b}{c\alpha + d}\right)\pi i\right),$$

and the summation runs over all pairs of coprime integers (c, d) that yield distinct values for the set $\{\underline{q}, -\underline{q}\}$. Moreover, for fixed c and d, a and b are any integral solutions of

$$ad - bc = 1. \tag{11.2.4}$$

Proof. Consider the integral

$$\frac{1}{2\pi i}\int_{H_m} \frac{f(z)}{z - q}\, dz, \tag{11.2.5}$$

where f is a function that satisfies the conditions specified in the theorem, H_m is a simple closed contour that is very close to (to be made more precise in the sequel) and inside (or perhaps touching) the unit circle, and q is fixed and inside H_m. By Cauchy's theorem, if $|q| < 1$,

$$\frac{1}{2\pi i}\int_{H_m} \frac{f(z)}{z - q}\, dz = f(q) + \Sigma_m \operatorname{Res}, \tag{11.2.6}$$

where $\Sigma_m \operatorname{Res}$ is the sum of the residues of $f(z)/(z - q)$ at the poles of f that are inside H_m. If we can show that the integral tends to zero as $m \to \infty$, then it will follow that $f(q) = -\Sigma\,\mathrm{Res}$, where the sum is over all residues of poles in the interior of $|z| = 1$.

First, we construct a contour H_m that will allow us to easily evaluate the integral. Our contour H_m is based on Farey fractions of order m. For basic properties of the Farey fractions of order m, which we denote by F_m, see, for example, [223, pp. 297–300]. For instance, if $h'/k' < h/k$ are two adjacent Farey fractions in F_m, then

$$hk' - h'k = 1; \tag{11.2.7}$$

also,

$$k' + k > m. \tag{11.2.8}$$

We now construct the desired contour. Suppose $h'/k' < h/k$ are adjacent Farey fractions in F_m. Construct the two semicircles in $\mathcal{H}$ that have the segments

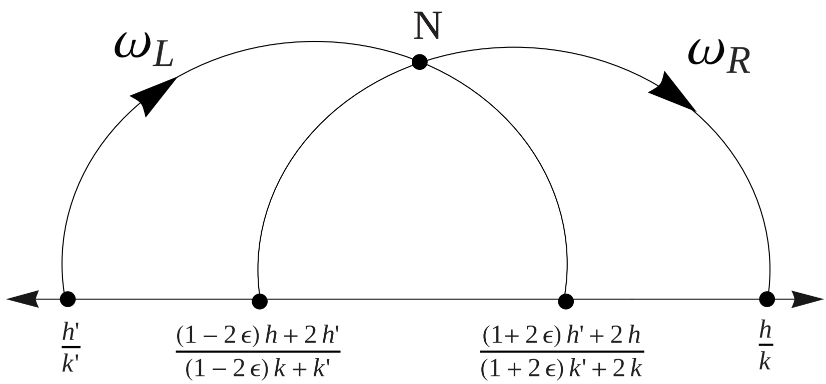
$$\left(\frac{h'}{k'}, \frac{(1+2\epsilon)h' + 2h}{(1+2\epsilon)k' + 2k}\right) \qquad \text{and} \qquad \left(\frac{(1-2\epsilon)h + 2h'}{(1-2\epsilon)k + 2k'}, \frac{h}{k}\right) \tag{11.2.9}$$

on the real axis as their diameters, where $\epsilon > 0$. The inequalities

$$\frac{h'}{k'} < \frac{h + 2h'}{k + 2k'} < \frac{h' + 2h}{k' + 2k} < \frac{h}{k}, \tag{11.2.10}$$

which follow from (11.2.7), imply that the circles intersect if we choose ϵ sufficiently small.

Let us say that N is their intersection point in the upper half-plane, ω_L is the arc from h'/k' to N, ω_R is the arc from N to h/k, and ω is the union of ω_L and ω_R.



Repeat the process for each adjacent pair of Farey fractions between 0 and 1. Thus we obtain a path from 0 to 1. Construct the mirror image of this path on the interval $[-1, 0]$, and call the entire contour (from -1 to 1) Ω_m.

If we regard Ω_m as being in the τ-plane, then the corresponding path in the q-plane, where $q = e^{\pi i \tau}$, is a simple closed contour that starts and ends at -1 and does not go outside the unit circle. This is our desired contour H_m. (Eventually, we shall let m go to ∞, so that H_m approaches the unit circle, as will be shown later.)

Now we show that each segment ω_L of the path Ω_m is the preimage of part of the left-hand boundary of the fundamental region P in $\mathcal{H}$ bounded by the circles $|\tau + 1| = 1$ and $|\tau| = 1$ and the lines $\operatorname{Re} \tau = -\frac{1}{2} - \epsilon$ and $\operatorname{Re} \tau = \frac{1}{2} - \epsilon$ under some modular transformation, and that each segment ω_R is the preimage of the right-hand boundary of P and a short line segment under some modular transformation. Later we use these properties to estimate f on the contour Ω_m.

If $h'/k' < h/k$ are adjacent Farey fractions, then

$$T_1(\tau) := \frac{k'\tau - h'}{-k\tau + h} \qquad \text{and} \qquad T_2(\tau) := \frac{k\tau - h}{k'\tau - h'} \tag{11.2.11}$$

are modular transformations because $hk' - h'k = 1$. These are the modular transformations to which we referred in the previous paragraph.

We first examine the transformation T_1. Note that under T_1, the preimages of the points $i\infty$, $\frac{1}{2} - \epsilon$, $-\frac{1}{2} - \epsilon$, 1, and -1 are

$$\frac{h}{k}, \quad \frac{(1-2\epsilon)h + 2h'}{(1-2\epsilon)k + 2k'}, \quad \frac{(1+2\epsilon)h - 2h'}{(1+2\epsilon)k - 2k'}, \quad \frac{h+h'}{k+k'}, \quad \text{and} \quad \frac{h-h'}{k-k'}, \tag{11.2.12}$$

respectively. Recall that, in the extended complex plane, modular transformations map the family of all circles and straight lines onto itself, and note from the definition of T_1 that $T_1(\bar{\tau}) = \overline{T_1(\tau)}$.

These imply that the preimages of the half-line $\operatorname{Re} \tau = \frac{1}{2} - \epsilon$, $\operatorname{Im} \tau \geq 0$, the half-line $\operatorname{Re} \tau = -\frac{1}{2} - \epsilon$, $\operatorname{Im} \tau \geq 0$, and the upper half of the unit circle are the semicircles in the upper half-plane $\mathcal{H}$ that have the segments on the real axis

$$\left(\frac{(1-2\epsilon)h + 2h'}{(1-2\epsilon)k + 2k'}, \frac{h}{k} \right), \quad \left(\frac{h}{k}, \frac{(1+2\epsilon)h - 2h'}{(1+2\epsilon)k - 2k'} \right), \quad \text{and} \quad \left(\frac{h'-h}{k'-k}, \frac{h'+h}{k'+k} \right) \tag{11.2.13}$$

as their diameters, respectively. Unless otherwise stated, the semicircles in this chapter are assumed to be in $\mathcal{H}$ with their diameters on the real axis.

Similarly, under the transformation T_2, the preimages are the semicircles that have the segments

$$\left(\frac{h'}{k'}, \frac{(1-2\epsilon)h' - 2h}{(1-2\epsilon)k' - 2k} \right), \quad \left(\frac{h'}{k'}, \frac{(1+2\epsilon)h' + 2h}{(1+2\epsilon)k' + 2k} \right), \quad \text{and} \quad \left(\frac{h'-h}{k'-k}, \frac{h'+h}{k'+k} \right) \tag{11.2.14}$$

as their diameters, respectively. Also, the preimage of the semicircle centered at -1 with radius 1 is the semicircle with the segment

$$\left(\frac{h+2h'}{k+2k'}, \frac{h}{k}\right) \tag{11.2.15}$$

as its diameter.

From (11.2.9) and (11.2.14), we see that T_2 maps ω_L into the half-line $\mathrm{Re}\ \tau = -\frac{1}{2} - \epsilon$, $\mathrm{Im}\ \tau \geq 0$. Also, by (11.2.9) and (11.2.13), we see that T_1 maps ω_R into the half-line $\mathrm{Re}\ \tau = \frac{1}{2} - \epsilon$, $\mathrm{Im}\ \tau \geq 0$.

Under T_1, the image of the left semicircle is the semicircle with the segment $(0, 2/(1+2\epsilon))$ as its diameter, while the image of the right semicircle is the half-line $\mathrm{Re}\ \tau = \frac{1}{2} - \epsilon$, $\mathrm{Im}\ \tau \geq 0$. The images intersect at the point

$$\tau_1 := \left(\frac{1}{2} - \epsilon\right) + i\sqrt{\frac{\frac{3}{4} - \frac{3}{2}\epsilon + \epsilon^2 - 2\epsilon^3}{1+2\epsilon}}; \tag{11.2.16}$$

so τ_1 must be the image of N. Thus the image of ω_R is the half-line

$$\mathrm{Re}\ \tau = \frac{1}{2} - \epsilon, \quad \mathrm{Im}\ \tau \geq \sqrt{\frac{\frac{3}{4} - \frac{3}{2}\epsilon + \epsilon^2 - 2\epsilon^3}{1+2\epsilon}}. \tag{11.2.17}$$

Similarly, under T_2, the images of the semicircles are the half-line $\mathrm{Re}\ \tau = -\frac{1}{2} - \epsilon$, $\mathrm{Im}\ \tau \geq 0$, and the semicircle with the segment $(-2/(1-2\epsilon), 0)$ as its diameter. These images intersect at

$$\tau_2 := \left(-\frac{1}{2} - \epsilon\right) + i\sqrt{\frac{\frac{3}{4} + \frac{3}{2}\epsilon + \epsilon^2 + 2\epsilon^3}{1-2\epsilon}},$$

the image of N. Thus the image of w_L is the half-line

$$\mathrm{Re}\ \tau = -\frac{1}{2} - \epsilon, \quad \mathrm{Im}\ \tau \geq \sqrt{\frac{\frac{3}{4} + \frac{3}{2}\epsilon + \epsilon^2 + 2\epsilon^3}{1-2\epsilon}}. \tag{11.2.18}$$

The intersection of the line $\mathrm{Re}\ \tau = \frac{1}{2} - \epsilon$ and the upper half of the unit circle is the point

$$\tau_3 := \left(\tfrac{1}{2} - \epsilon\right) + i\sqrt{\tfrac{3}{4} + \epsilon - \epsilon^2}, \tag{11.2.19}$$

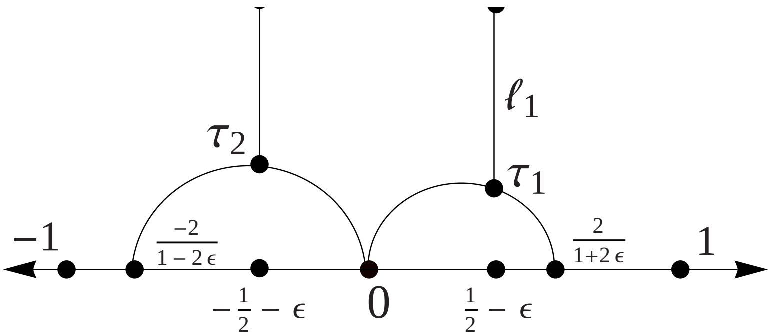
which is the lowermost point on the right-hand boundary of P. Similarly, the lowermost point on the left-hand boundary of P is

$$\tau_4 := \left(-\tfrac{1}{2} - \epsilon\right) + i\sqrt{\tfrac{3}{4} + \epsilon - \epsilon^2}.$$

Note that for $\epsilon > 0$,

$$\sqrt{\frac{\frac{3}{4} - \frac{3}{2}\epsilon + \epsilon^2 - 2\epsilon^3}{1+2\epsilon}} < \sqrt{\frac{3}{4} + \epsilon - \epsilon^2}, \tag{11.2.20}$$

because



$$\tfrac{3}{4} - \tfrac{3}{2}\epsilon + \epsilon^2 - 2\epsilon^3 < \left(\tfrac{3}{4} + \epsilon - \epsilon^2\right)(1+2\epsilon) = \tfrac{3}{4} + \tfrac{5}{2}\epsilon + \epsilon^2 - 2\epsilon^3.$$

Similarly,

$$\sqrt{\frac{3}{4} + \epsilon - \epsilon^2} < \sqrt{\frac{\frac{3}{4} + \frac{3}{2}\epsilon + \epsilon^2 + 2\epsilon^3}{1 - 2\epsilon}}\,. \tag{11.2.21}$$

By (11.2.17) and (11.2.18), inequalities (11.2.20) and (11.2.21) imply that T_2 maps ω_L onto part of the left-hand boundary of P, while the transformation T_1 maps ω_R into the right-hand boundary of P and a line segment below the right-hand boundary of P. We denote by ℓ_1 the line segment that has τ_1 and τ_3 as its endpoints. Note that by (11.2.16) and (11.2.19), the length of ℓ_1 tends to 0 as ϵ tends to 0.

Also, if ω_L, ω_R, and ω are curves in the τ-plane, then let $\mathcal{C}_L$, $\mathcal{C}_R$, and $\mathcal{C}$ be the corresponding arcs in the q-plane, where $q = e^{\pi i \tau}$.

Next we obtain an estimate for the integral of $f(z)/(z-q)$ over $\mathcal{C}_R$, one segment of H_m. We then use this to obtain an estimate of the integral over all of H_m. We begin by finding an upper bound for $|f|$ on $\mathcal{C}_R$ (which is equivalent to finding an upper bound for $|\varphi|$ on ω_R).

Recall that $f(q) = f(e^{\pi i \tau}) = \varphi(\tau)$ is analytic at $q = 0$. So $\lim_{q\to 0} f(q)$ exists and $\lim_{\tau\to i\infty}\varphi(\tau)$ exists. Recall also that φ has only one pole in each fundamental region. So φ has only one pole in P_1 and one pole in P_4. Thus we see that by our choice of ϵ we can avoid having a pole on the left- or right-hand boundary of P. Since $\lim_{\tau\to i\infty}\varphi(\tau)$ exists, we know that $|\varphi(\tau)|$ is bounded on the right- and left-hand boundaries of P.

Now consider the line segment ℓ_1. If ϵ is sufficiently small, then ℓ_1 is in at most two fundamental regions, P_2 and P_3. Since each of these regions has only one pole of φ, we can avoid having a pole on ℓ_1 by choosing ϵ carefully. Thus, $|\varphi(\tau)|$ is bounded on ℓ_1 as well. So we can say that

$$|\varphi| \le M \tag{11.2.22}$$

on ℓ_1 and the right- and left-hand boundaries of P, where M is some absolute constant. Note that M is independent of m, where m is the order of F_m.

By the functional equation (11.2.1),

$$|\varphi(\tau)| = \left|\varphi\left(\frac{k'\tau - h'}{-k\tau + h}\right)\right| |-k\tau + h|^n. \tag{11.2.23}$$

If τ is on w_R, then, by (11.2.22) and (11.2.23),

$$|\varphi(\tau)| \le M|-k\tau + h|^n = Mk^n|\tau - h/k|^n, \tag{11.2.24}$$

because T_1 maps ω_R onto ℓ_1 and the right-hand boundary of P. For τ on ω_{R_1}, the quantity $|\tau - h/k|$ is maximized when $\tau = N$. We need to estimate $|N - h/k|$. By (11.2.16), N is the preimage of τ_1 under the transformation T_1, and so

$$N = T_1^{-1}(\tau_1) = \frac{h\tau_1 + h'}{k\tau_1 + k'} \; .$$

Thus, by (11.2.7) and (11.2.16),

$$\begin{aligned}
\left|N - \frac{h}{k}\right| &= \frac{1}{|kk' + k^2\tau_1|} \\
&= \frac{1}{\left|kk' + k^2\left(\dfrac{1}{2} - \epsilon\right) + ik^2\sqrt{\dfrac{\frac{3}{4} - \frac{3}{2}\epsilon + \epsilon^2 - 2\epsilon^3}{1 + 2\epsilon}}\right|} \\
&= \frac{1}{\sqrt{\left(kk' + k^2(\frac{1}{2} - \epsilon)\right)^2 + k^4\left(\dfrac{\frac{3}{4} - \frac{3}{2}\epsilon + \epsilon^2 - 2\epsilon^3}{1 + 2\epsilon}\right)}} \\
&< \frac{1}{\sqrt{\left(\frac{1}{2}kk' + \frac{1}{4}k^2\right)^2 + \frac{3}{16}k^4}} \\
&= \frac{2}{k\sqrt{k^2 + kk' + k'^2}},
\end{aligned} \tag{11.2.25}$$

for ϵ sufficiently small.

On ω_R, by (11.2.24) and (11.2.25),

$$|\varphi(\tau)| < Mk^n\left(\frac{2}{k\sqrt{k^2 + kk' + k'^2}}\right)^n = \frac{2^n M}{(k^2 + kk' + k'^2)^{n/2}} \; . \tag{11.2.26}$$

Thus, we have obtained a bound for $|f|$ on $\mathcal{C}_R$.

If q is fixed and inside H_m, then

$$\left|\frac{f(z)}{z - q}\right| < \frac{M_1}{(k^2 + kk' + k'^2)^{n/2}} \tag{11.2.27}$$

for z on $\mathcal{C}_R$, where M_1 is some constant that depends on q and n.

Now we estimate the length of $\mathcal{C}_R$. We first calculate the arc length of ω_R, then make the change of variable $q = \exp(\pi i \tau)$, and lastly estimate $\mathcal{C}_R$. From (11.2.9) and (11.2.10), we see that the length of ω_R is less than

$$\frac{1}{2}\pi\left(\frac{h}{k} - \frac{h'}{k'}\right) = \frac{1}{2}\pi\left(\frac{1}{kk'}\right)$$

by (11.2.7). Because

$$\left|\frac{dq}{d\tau}\right| = |\pi i e^{\pi i \tau}| = |\pi i q| \leq \pi$$

on this arc, the length of $\mathcal{C}_R$ is less than

$$\frac{\pi^2}{2}\frac{1}{kk'}. \tag{11.2.28}$$

Thus, by (11.2.27) and (11.2.28),

$$\left|\int_{\mathcal{C}_R} \frac{f(z)}{z-q}\,dz\right| < \frac{M_2}{kk'(k^2+kk'+k'^2)^{n/2}}\,, \tag{11.2.29}$$

where M_2 is some constant that depends on q and n. Using the transformation T_2, we can obtain an identical result for the integral over $\mathcal{C}_L$.

So far we have examined only the portion of Ω_m in the right half-plane and the portion of H_m in the upper half-plane. Because Ω_m is symmetric about 0, it follows that H_m is symmetric about the real axis. By applying the reasoning above to the interval $[-h/k, -h'/k']$, we obtain identical results for the arcs in the lower half-plane that are mirror images of the arcs $\mathcal{C}_L$ and $\mathcal{C}_R$ in the upper half-plane, which we have just analyzed.

We are ready to estimate the integral over H_m. Using (11.2.29), we obtain the inequality

$$\left|\int_{H_m} \frac{f(z)}{z-q}\right| < 4M_2 \sum_{\left(\frac{h'}{k'},\frac{h}{k}\right)} \frac{1}{kk'(k^2+kk'+k'^2)^{n/2}}\,, \tag{11.2.30}$$

where the summation runs over all adjacent pairs of Farey fractions in F_m, and M_2 is, of course, independent of m. We want to show that the right-hand side tends to zero as m tends to ∞.

To that end, observe that

$$\begin{aligned}\sum_{\left(\frac{h'}{k'},\frac{h}{k}\right)} \frac{1}{kk'(k^2+kk'+k'^2)^{n/2}} &< \sum_{\left(\frac{h'}{k'},\frac{h}{k}\right)} \frac{1}{kk'\left(\frac{k+k'}{2}\right)^n} \\ &< \left(\frac{2}{m}\right)^n \sum_{\left(\frac{h'}{k'},\frac{h}{k}\right)} \left(\frac{h}{k} - \frac{h'}{k'}\right) = \left(\frac{2}{m}\right)^n.\end{aligned} \tag{11.2.31}$$

Therefore, by (11.2.30) and (11.2.31),

$$\lim_{m\to\infty} \frac{1}{2\pi i} \int_{H_m} \frac{f(z)}{z-q} = 0. \tag{11.2.32}$$

Note that for a typical arc $\mathcal{C}$, the maximum distance from the arc to the unit circle is less than the length of $\mathcal{C}$. From our calculation of the maximum possible length of $\mathcal{C}_R$ in (11.2.28), we see that the length of $\mathcal{C}$ is less than

$$\frac{\pi^2}{kk'} < \frac{\pi^2}{k(m-k)} \leq \frac{\pi^2}{m-1}$$

if $k < m$, and it is less than

$$\frac{\pi^2}{kk'} \leq \frac{\pi^2}{m}$$

if $k = m$. So as m tends to ∞, H_m approaches the unit circle uniformly from the inside. Therefore all the poles of $f(z)$ that are inside the unit circle are eventually inside H_m. By (11.2.6) and (11.2.32),

$$f(q) = -\sum \text{Res}, \tag{11.2.33}$$

where $\Sigma\,\text{Res}$ is the sum of the residues of $f(z)/(z-q)$ at the poles of f that are inside the unit circle. We next determine these poles.

Recall that $\varphi(\tau)$ has only one pole in P_1, a simple pole at $\tau = \alpha$, with residue A. By the functional equation (11.2.1), the only poles of $\varphi(\tau)$ in the upper half-plane are at the points $\tau = (a\alpha + b)/(c\alpha + d)$, where a, b, c, $d \in \mathbb{Z}$ and $ad - bc = 1$.

If c and d are fixed, and (a, b) is one solution to $ad - bc = 1$, then the complete set of solutions is $\{(a + mc, b + md) : m \in \mathbb{Z}\}$. Each of these solutions produces a distinct pole of $\varphi(\tau)$. However, this set yields only two distinct poles of $f(q)$, namely,

$$q = \pm \exp\left(\pi i \frac{a\alpha + b}{c\alpha + d}\right), \tag{11.2.34}$$

because

$$\exp\left(\pi i \frac{(a + mc)\alpha + (b + md)}{c\alpha + d}\right) = (-1)^m \exp\left(\pi i \frac{a\alpha + b}{c\alpha + d}\right).$$

If we let (c, d) range over all pairs of coprime integers, then the two expressions in (11.2.34) will eventually take on as their values each of the poles of f inside the unit circle.

However, as we will see later when applying the theorem, it is possible that different pairs (c, d) may produce the same poles of $f(q)$. In our applications, we need to be careful when calculating the sum $\Sigma\,\text{Res}$ so that we do not count a residue of the same pole twice.

We now calculate the residues of $f(q)$ at its poles. If we let $T := (a\tau + b)/(c\tau + d)$, then $\tau = (dT - b)/(-cT + a)$. When we substitute T for τ and find a common denominator, we find that

$$\begin{aligned}
A = \operatorname{Res}(\varphi(\tau), \alpha) &= \lim_{\tau\to\alpha}(c\tau+d)^n\varphi\left(\frac{a\tau+b}{c\tau+d}\right)(\tau-\alpha)\\
&= (c\alpha+d)^n \lim_{\tau\to\alpha}\varphi(T)\left(\frac{dT-b}{-cT+a}-\alpha\right)\\
&= (c\alpha+d)^n \lim_{T\to(a\alpha+b)/(c\alpha+d)}\varphi(T)\left(\frac{(c\alpha+d)T-(a\alpha+b)}{-cT+a}\right)\\
&= (c\alpha+d)^n \lim_{T\to(a\alpha+b)/(c\alpha+d)}\varphi(T)\left(\frac{T-\dfrac{a\alpha+b}{c\alpha+d}}{-c\left(\dfrac{a\alpha+b}{c\alpha+d}\right)+a}\right)(c\alpha+d)\\
&= (c\alpha+d)^{n+2} \lim_{T\to(a\alpha+b)/(c\alpha+d)}\varphi(T)\left(T-\frac{a\alpha+b}{c\alpha+d}\right). \qquad (11.2.35)
\end{aligned}$$

Note that the right-hand side of (11.2.35) is $(c\alpha+d)^{n+2}$ times the residue of $\varphi(\tau)$ at $\tau = (a\alpha+b)/(c\alpha+d)$. Hence,

$$\operatorname{Res}\left(\varphi(\tau), \frac{a\alpha+b}{c\alpha+d}\right) = \frac{A}{(c\alpha+d)^{n+2}}. \qquad (11.2.36)$$

Using (11.2.36) and the fact that

$$\left.\frac{dq}{d\tau}\right|_{\tau=\frac{a\alpha+b}{c\alpha+d}} = \pi i \exp\left(\pi i\frac{a\alpha+b}{c\alpha+d}\right),$$

we find that

$$\operatorname{Res}\left(f(q), \pm\exp\left(\pi i\frac{a\alpha+b}{c\alpha+d}\right)\right) = \pm\frac{\pi i A\exp\left(\pi i\dfrac{a\alpha+b}{c\alpha+d}\right)}{(c\alpha+d)^{n+2}} = \pm\frac{\pi i A\underline{q}}{(c\alpha+d)^{n+2}}, \qquad (11.2.37)$$

where

$$\underline{q} = \exp\left(\pi i\frac{a\alpha+b}{c\alpha+d}\right).$$

When we use (11.2.37) to evaluate the sum $-\sum\operatorname{Res}$ in (11.2.33), we find that (11.2.33) becomes

$$\begin{aligned}
f(q) &= -\sum\left(\frac{\pi i A\underline{q}}{(c\alpha+d)^{n+2}}\frac{1}{\underline{q}-q} + \frac{-\pi i A\underline{q}}{(c\alpha+d)^{n+2}}\frac{1}{-\underline{q}-q}\right)\\
&= -2\pi i A\sum\frac{1}{(c\alpha+d)^{n+2}}\frac{1}{1-(q/\underline{q})^2}, \qquad (11.2.38)
\end{aligned}$$

where the summation runs over all pairs of coprime integers (c,d) that yield distinct values for $\pm\underline{q}$ (the poles of $f(q)$), and for fixed (c,d), (a,b) is any integral solution to $ad-bc=1$. Thus the proof of the theorem is complete for $|q|<1$.

If $|q|>1$, the proof is the same except that now there is not a pole of $f(q)$ inside the unit circle. Thus, the term $f(q)$ in (11.2.2) does not appear, and so we arrive at (11.2.3) instead. □

11.3 The Coefficients of $1/Q(q)$

As we noted earlier, Hardy and Ramanujan used Theorem 11.2.1 to prove a formula for the coefficients in the power series expansion of the reciprocal of the Eisenstein series $E_6(\tau)=R(q)$. In this section we prove one of the analogous results found with the publication of the lost notebook [244, pp. 102–104], [74, pp. 179–182]. These three pages are apparently taken from one of Ramanujan's letters to Hardy in 1918, but the pages are undated and bear no salutation.

Let $K=\mathbb{Q}(\sqrt{-3})$. The algebraic integral domain $\mathfrak{O}_K=\mathbb{Z}[\zeta]=\mathbb{Z}\oplus\zeta\mathbb{Z}$, where $\zeta=\exp(2\pi i/3)$, is a principal ideal domain. If (c,d) is a pair of coprime integers that is a solution to the equation

$$\lambda=c^2-cd+d^2, \tag{11.3.1}$$

where λ is a certain fixed positive integer, then

$$\pm(c,d),\quad \pm(d,c),\quad \pm(c-d,c),\quad \pm(c,c-d),\quad \pm(d,d-c),\quad \pm(c-d,-d) \tag{11.3.2}$$

are solutions as well. To see this, set $\alpha=c+d\zeta$ and let $\mathfrak{A}=(c+d\zeta)$ be an ideal. Then, if N denotes the norm of $\mathfrak{A}$, we see that

$$\lambda=N(\mathfrak{A})=\mathfrak{A}\bar{\mathfrak{A}}=(c+d\zeta)(c+d\bar{\zeta}),$$

where $\bar{\zeta}=\zeta^2$. The unit group in $\mathfrak{O}_K$ is $U:=\{\pm1,\pm\zeta,\pm\zeta^2\}$. It follows that

$$\begin{aligned}\mathfrak{A}&=(\alpha)=(-\alpha)=(\alpha\zeta)=(-\alpha\zeta)=(\alpha\zeta^2)=(-\alpha\zeta^2),\\ \bar{\mathfrak{A}}&=(\bar{\alpha})=(-\bar{\alpha})=(\bar{\alpha}\zeta)=(-\bar{\alpha}\zeta)=(\bar{\alpha}\zeta^2)=(-\bar{\alpha}\zeta^2).\end{aligned}$$

Hence, one solution generates twelve solutions. For example, if

$$\mathfrak{A}=(\alpha\zeta)=(c\zeta+d\zeta^2)=(c\zeta+d(-1-\zeta))=(-d+(c-d)\zeta),$$

then $(-d,c-d)$ is a solution if (c,d) is a solution. Note that if λ is a prime, then the twelve solutions are the only solutions, but if λ is composite, there may be many solutions.

Definition 11.3.1. *We say that two solutions* (c_1, d_1) *and* (c_2, d_2) *to the equation* (11.3.1) *are distinct if they do not simultaneously belong to the same set of solutions in* (11.3.2).

Note that each of the solutions in (11.3.2) has an element from two of the three sets $\{\pm c\}$, $\{\pm d\}$, and $\{\pm(c-d)\}$. Thus if two solutions simultaneously belong to (11.3.2), then they have an element in common at least in absolute value. It follows that (c_1, d_1) and (c_2, d_2) are distinct solutions to (11.3.1) if and only if

$$c_2, d_2 \notin \{\pm c_1, \pm d_1\}. \tag{11.3.3}$$

It is well known, see, e.g., the text by Niven, Zuckerman, and Montgomery [223, p. 176], that the integers λ that can be represented in the form of $\lambda = c^2 - cd + d^2$, with c and d coprime, are integers of the form

$$\lambda = 3^a \prod_{j=1}^{r} p_j^{a_j}, \tag{11.3.4}$$

where $a = 0$ or 1, p_j is a prime of the form $6m+1$, and a_j is a nonnegative integer, $1 \le j \le r$.

Entry 11.3.1 (p. 103). *Recall that* $Q(q)$ *and* $R(q)$ *are defined by* (11.1.2) *and* (11.1.3), *respectively. Let*

$$\frac{1}{Q(q^2)} = \sum_{n=0}^{\infty} \beta_n q^{2n}, \qquad |q| < q_0,$$

and

$$G := R(e^{2\pi i\rho}) = 1 - 504 \sum_{k=1}^{\infty} \frac{(-1)^k k^5}{e^{k\pi\sqrt{3}} - (-1)^k} = 2.8815\ldots, \tag{11.3.5}$$

where $\rho := -1/2 + i\sqrt{3}/2$. *Then*

$$\begin{aligned}\beta_n = (-1)^n \frac{3}{G} \Biggl\{ & e^{n\pi\sqrt{3}} - \frac{e^{n\pi\sqrt{3}/3}}{3^3} + \frac{2\cos\left(\dfrac{2\pi n}{7} - 6\arctan(-3\sqrt{3})\right)}{7^3} e^{n\pi\sqrt{3}/7} \\ & + \frac{2\cos\left(\dfrac{6\pi n}{13} - 6\arctan(-2\sqrt{3})\right) e^{n\pi\sqrt{3}/13}}{13^3} + \cdots \Biggr\} \\ = (-1)^n \frac{3}{G} & \sum_{(\lambda)} \frac{h_\lambda(n)}{\lambda^3} e^{n\pi\sqrt{3}/\lambda}. \end{aligned} \tag{11.3.6}$$

Here λ *runs over the integers of the form* (11.3.4),

$$h_1(n) = 1, \qquad h_3(n) = -1, \tag{11.3.7}$$

and, for $\lambda \geq 7$,

$$h_\lambda(n) = 2\sum_{(c,d)} \cos\left((ad+bc-2ac-2bd+\lambda)\frac{n\pi}{\lambda} - 6\arctan\left(\frac{c\sqrt{3}}{2d-c}\right)\right), \tag{11.3.8}$$

where the sum is over all pairs (c,d), *where* (c,d) *is a distinct solution to* $\lambda = c^2 - cd + d^2$ *and, for fixed* (c,d), (a,b) *is any solution to* $ad - bc = 1$. *Also, distinct solutions* (c,d) *to* $\lambda = c^2 - cd + d^2$ *give rise to distinct terms in the sum in* (11.3.6). *Furthermore, if* $n < 0$, *the sum on the far right side of* (11.3.6) *equals* 0.

Proof. We apply Theorem 11.2.1 to the function $1/Q(q^2)$. Then $\varphi(\tau) = 1/Q(e^{2\pi i\tau}) = 1/E_4(\tau)$. Since the Eisenstein series $E_{2j}(\tau)$ is a modular form of degree $-2j$ [255, p. 50], $\varphi(\tau)$ satisfies the functional equation (11.2.1) with $n = 4$, i.e.,

$$\varphi(\tau) = \varphi\left(\frac{a\varphi + b}{c\tau + d}\right)(c\tau + d)^4. \tag{11.3.9}$$

The function $\varphi(\tau)$ has only one pole in P_1, a simple pole at $\tau = -\frac{1}{2} + i\frac{\sqrt{3}}{2} =: \rho$ [246, p. 198]. Thus, in (11.2.2), we have

$$\alpha = \rho. \tag{11.3.10}$$

Clearly $\varphi(\tau)$ is meromorphic in $\mathcal{H}$, which implies that $f(q)$ is meromorphic in the unit disk.

We now calculate $A = \operatorname{Res}(\varphi, \rho)$ by calculating the corresponding residue, $\operatorname{Res}(f, e^{\pi i\rho})$.

Suppose that a function $F(q)$ has a simple pole at $q = q_1$. Expanding $F(q)$ into its Laurent series about $q = q_1$, we can easily see that

$$\operatorname{Res}(F, q_1) = \left.\frac{1}{d(1/F(q))/dq}\right|_{q=q_1}. \tag{11.3.11}$$

By (11.3.11), (11.1.2), and (11.1.8),

$$\begin{aligned}\operatorname{Res}(f, e^{\pi i\rho}) = \left.\frac{1}{d(Q(q^2))/dq}\right|_{q=e^{\pi i\rho}} &= \frac{3}{2}\left.\frac{q}{P(q^2)Q(q^2) - R(q^2)}\right|_{q=e^{\pi i\rho}} \\ &= -\frac{3}{2}\frac{e^{\pi i\rho}}{R(e^{2\pi i\rho})} = -\frac{3e^{\pi i\rho}}{2G},\end{aligned}$$

where G is given by (11.3.5).

If we apply (11.2.37) with $\alpha = \rho$ and $(a,b,c,d) = (1,0,0,1)$, then we deduce that

$$\pi i A e^{\pi i\rho} = -\frac{3e^{\pi i\rho}}{2G},$$

or

$$A = -\frac{3}{2\pi i G}. \tag{11.3.12}$$

By (11.2.2), (11.3.9), (11.3.10), and (11.3.12), we find that

$$f(q) = \frac{3}{G} \sum_{(c,d)} \frac{1}{(c\rho + d)^6} \frac{1}{1 - (q/\underline{q})^2}, \tag{11.3.13}$$

where

$$\underline{q} = \exp\left(\pi i \left(\frac{a\rho + b}{c\rho + d}\right)\right), \tag{11.3.14}$$

and where the summation runs over all pairs of coprime integers (c, d) that produce distinct values for the set $\{\underline{q}, -\underline{q}\}$, and (a, b) is any integral solution to $ad - bc = 1$.

By (11.2.34), each pair (c, d) leads to exactly two distinct poles of f in the unit circle, $\underline{q}$ and $-\underline{q}$, but it is possible that different pairs may lead to the same poles, so we need to be careful that we do not count the same pole twice in the summation.

Thus, two tasks remain: find the values of (c, d) over which the summation runs, and compare the coefficients of q^n on both sides of (11.3.13).

First, if $\lambda = c^2 - cd + d^2$, then

$$\begin{aligned} \frac{a\rho + b}{c\rho + d} &= \frac{(a\rho + b)(c\rho^2 + d)}{\lambda} = \frac{ac + bd + (ad - bc)\rho + bc(\rho + \rho^2)}{\lambda} \\ &= \frac{ac + bd - bc - \frac{1}{2} + \frac{\sqrt{3}}{2} i}{\lambda}. \end{aligned}$$

So,

$$\underline{q} = \exp\left(\frac{-\pi\sqrt{3}}{2\lambda}\right) \exp\left(\frac{\pi i}{\lambda}\left(ac + bd - \frac{1}{2}ad - \frac{1}{2}bc\right)\right). \tag{11.3.15}$$

If two pairs (c_3, d_3) and (c_4, d_4) produce distinct values of λ, i.e., $c_3^2 - c_3 d_3 + d_3^2 = \lambda_3 \neq \lambda_4 = c_4^2 - c_4 d_4 + d_4^2$, then those pairs lead to distinct values for the set $\{\pm\underline{q}\}$, say $\{\pm\underline{q}_3\}$ and $\{\pm\underline{q}_4\}$, i.e.,

$$\{\pm\underline{q}_3\} \cap \{\pm\underline{q}_4\} = \emptyset. \tag{11.3.16}$$

We now consider the case in which different values of (c, d) produce the same values of λ. As we saw in (11.3.2), each solution (c, d) to $\lambda = c^2 - cd + d^2$ generates a total of twelve solutions. If $\lambda = 1$ or $\lambda = 3$, then only six of these twelve are different solutions. If $\lambda \geq 7$, then the twelve solutions are all different.

Suppose that the solution (c, d) leads to $\{\pm\underline{q}_5\}$, say. Then $(-c, -d)$ also leads to $\{\pm\underline{q}_5\}$, while (d, c) and $(c, c-d)$ both lead to $\{\pm\bar{\underline{q}}_5\}$. Since these three basic transformations lead to either $\{\pm \underline{q}_5\}$ or $\{\pm \bar{\underline{q}}_5\}$, it follows that (c,d) and the eleven corresponding solutions of (11.3.2) yield a set of only four different poles, namely,

$$\{\pm \underline{q}_5, \pm \bar{\underline{q}}_5\}. \tag{11.3.17}$$

This would be a set of only two poles if $\underline{q}_5$ were real or purely imaginary. We prove in Lemmas 11.3.1, 11.3.2, and 11.3.3 at the end of this section that $\underline{q}$ can never be real, and that it is purely imaginary only when $\lambda = 1$ or $\lambda = 3$. Thus, (11.3.17) is valid only for $\lambda \geq 7$. If $\lambda = 1$ or $\lambda = 3$, then the solutions of (11.3.2) produce a set of only two poles (for each value of λ),

$$\{\pm \underline{q}_5\}. \tag{11.3.18}$$

Lastly, suppose that (c_1, d_1) and (c_2, d_2) are distinct solutions to equation (11.3.1). In Lemma 11.3.4 at the end of the section, we prove that each distinct solution (together with its eleven corresponding solutions in (11.3.2)) yields four distinct poles, i.e.,

$$\{\pm \underline{q}_1, \pm \bar{\underline{q}}_1\} \cap \{\pm \underline{q}_2, \pm \bar{\underline{q}}_2\} = \emptyset, \tag{11.3.19}$$

where the bracketed sets correspond to (c_1, d_1) and (c_2, d_2), respectively.

We can now express the right-hand side of (11.3.13) not as a sum over pairs (c,d), but as a sum over λ and over distinct pairs (c,d).

From (11.3.13) and (11.3.16)–(11.3.19),

$$f(q) = \frac{3}{G}\left\{\sum_{\substack{(\lambda)\\ \lambda \leq 3}} \frac{1}{(c\rho+d)^6}\,\frac{1}{1-(q/\underline{q})^2}\right. \tag{11.3.20}$$
$$\left. + \sum_{\substack{(\lambda)\\ \lambda > 3}} \left(\frac{1}{(c\rho+d)^6}\,\frac{1}{1-(q/\underline{q})^2} + \frac{1}{(d\rho+c)^6}\,\frac{1}{1-(q/\bar{\underline{q}})^2}\right)\right\},$$

where λ runs over all integers of the form (11.3.4), and where, for each fixed λ, the sum is also over all distinct pairs (c,d).

For $\lambda = 1$, with $(a,b,c,d) = (1,0,0,1)$, by (11.3.15),

$$\underline{q} = \exp\left(-\pi\sqrt{3}/2\right)\exp(-\pi i/2) = -ie^{-\pi\sqrt{3}/2}.$$

Thus,

$$\frac{1}{(c\rho+d)^6}\,\frac{1}{1-(q/\underline{q})^2} = \frac{1}{1+e^{\pi\sqrt{3}}q^2}. \tag{11.3.21}$$

Similarly, for $\lambda = 3$, with $(a,b,c,d) = (1,1,1,2)$, by (11.3.15),

$$\underline{q} = \exp\left(\frac{-\pi\sqrt{3}}{6}\right)\exp\left(\frac{\pi i}{3}\left(\frac{3}{2}\right)\right) = i\exp\left(\frac{-\pi\sqrt{3}}{6}\right).$$

Thus,

$$\frac{1}{(c\rho+d)^6}\ \frac{1}{1-(q/\underline{q})^2}=-\frac{1}{27}\ \frac{1}{1+e^{\pi\sqrt{3}/3}q^2}. \tag{11.3.22}$$

By (11.3.21) and (11.3.22), equality (11.3.20) becomes

$$\begin{aligned}
f(q) &= \frac{3}{G}\left\{\frac{1}{1+e^{\pi\sqrt{3}}q^2}-\frac{1}{27}\frac{1}{1+e^{\pi\sqrt{3}/3}q^2}\right.\\
&\qquad\left.+\sum_{\substack{(\lambda)\\ \lambda>3}}\left(\frac{1}{(c\rho+d)^6}\ \frac{1}{1-(q/\underline{q})^2}+\frac{1}{(d\rho+c)^6}\ \frac{1}{1-(q/\underline{\bar{q}})^2}\right)\right\}\\
&=\frac{3}{G}\left\{\sum_{n=0}^{\infty}(-1)^n e^{n\pi\sqrt{3}}q^{2n}-\frac{1}{27}\sum_{n=0}^{\infty}(-1)^n e^{n\pi\sqrt{3}/3}q^{2n}\right.\\
&\qquad\left.+\sum_{\substack{(\lambda)\\ \lambda>3}}\left(\frac{1}{(c\rho+d)^6}\sum_{n=0}^{\infty}\underline{q}^{-2n}q^{2n}+\frac{1}{(d\rho+c)^6}\ \sum_{n=0}^{\infty}\underline{\bar{q}}^{-2n}q^{2n}\right)\right\}\\
&=\sum_{n=0}^{\infty}\beta_n q^{2n},
\end{aligned} \tag{11.3.23}$$

where $|q|<e^{-\pi\sqrt{3}/2}$ and

$$\begin{aligned}
\beta_n &= (-1)^n\frac{3}{G}\left(e^{n\pi\sqrt{3}}-\frac{e^{n\pi\sqrt{3}/3}}{3^3}\right)\\
&\qquad+\frac{3}{G}\sum_{\substack{(\lambda)\\ \lambda>3}}\left(\frac{1}{(c\rho+d)^6}\underline{q}^{-2n}+\frac{1}{(d\rho+c)^6}\underline{\bar{q}}^{-2n}\right).
\end{aligned} \tag{11.3.24}$$

We now show that

$$\frac{1}{(c\rho+d)^6}=\overline{\left(\frac{1}{(d\rho+c)^6}\right)}, \tag{11.3.25}$$

and then we use this to express the sum in (11.3.24) more explicitly. By an elementary calculation, we find that

$$\frac{1}{(c\rho+d)^6}=\frac{1}{\lambda^3}\exp\left(-6i\arctan\left(\frac{c\sqrt{3}}{2d-c}\right)\right), \tag{11.3.26}$$

and similarly,

$$\frac{1}{(d\rho+c)^6}=\frac{1}{\lambda^3}\exp\left(-6i\arctan\left(\frac{d\sqrt{3}}{2c-d}\right)\right). \tag{11.3.27}$$

Note, however, that

$$\tan\left(\arctan\left(\frac{c\sqrt{3}}{2d-c}\right)+\arctan\left(\frac{d\sqrt{3}}{2c-d}\right)\right)$$
$$=\frac{\dfrac{c\sqrt{3}}{2d-c}+\dfrac{d\sqrt{3}}{2c-d}}{1-\left(\dfrac{c\sqrt{3}}{2d-c}\right)\left(\dfrac{d\sqrt{3}}{2c-d}\right)}=-\sqrt{3}.$$

Hence,

$$\arctan\left(\frac{c\sqrt{3}}{2d-c}\right)+\arctan\left(\frac{d\sqrt{3}}{2c-d}\right)=m\pi-\frac{\pi}{3}, \tag{11.3.28}$$

where m is some integer. Thus from (11.3.26), (11.3.27), and (11.3.28),

$$\frac{1}{(c\rho+d)^6}\,\frac{1}{(d\rho+c)^6}=\frac{1}{\lambda^6}\exp\left(-6i\left(m\pi-\frac{\pi}{3}\right)\right)=\frac{1}{\lambda^6}, \tag{11.3.29}$$

which is, of course, real. Since $\{(c\rho+d)^6(d\rho+c)^6\}^{-1}$ is real, (11.3.25) follows. From (11.3.15),

$$\underline{q}^{-2n}=(-1)^n\exp\left(\frac{n\pi\sqrt{3}}{\lambda}\right)\exp\left(\frac{n\pi i}{\lambda}\left(ad+bc-2ac-2bd+\lambda\right)\right). \tag{11.3.30}$$

Thus, by (11.3.29), (11.3.26), and (11.3.30), each summand in the sum of (11.3.24) is

$$2\mathrm{Re}\left(\frac{\underline{q}^{-2n}}{(c\rho+d)^6}\right)=2(-1)^n e^{n\pi\sqrt{3}/\lambda}$$
$$\times\,\mathrm{Re}\left(\frac{\exp\left(\dfrac{n\pi i}{\lambda}(ad+bc-2ac-2bd+\lambda)\right)\exp\left(-6i\arctan\left(\dfrac{c\sqrt{3}}{2d-c}\right)\right)}{\lambda^3}\right)$$
$$=\frac{(-1)^n}{\lambda^3}2\cos\left((ad+bc-2ac-2bd+\lambda)\frac{n\pi}{\lambda}-6\arctan\left(\frac{c\sqrt{3}}{2d-c}\right)\right)e^{n\pi\sqrt{3}/\lambda}. \tag{11.3.31}$$

From (11.3.24) and (11.3.31), the coefficient of q^{2n} in the power series expansion of $f(q)$ is

$$\beta_n = (-1)^n \frac{3}{G} \left\{ e^{n\pi\sqrt{3}} - \frac{e^{n\pi\sqrt{3}/3}}{3^3} \right.$$

$$\left. + \sum_{\substack{(\lambda) \\ \lambda>3}} \frac{2\cos\left((ad+bc-2ac-2bd+\lambda)\dfrac{n\pi}{\lambda} - 6\arctan\left(\dfrac{c\sqrt{3}}{2d-c}\right)\right)}{\lambda^3} e^{n\pi\sqrt{3}/\lambda} \right\}$$

$$= (-1)^n \frac{3}{G} \sum_{(\lambda)} \frac{h_\lambda(n)}{\lambda^3} e^{n\pi\sqrt{3}/\lambda}, \tag{11.3.32}$$

where $h_\lambda(n)$ is defined in (11.3.7) and (11.3.8). This proves (11.3.6).

To obtain the displayed terms in the expansion (11.3.6), we choose $(a,b,c,d) = (1,0,3,1)$ for $\lambda = 7$ and $(a,b,c,d) = (1,0,4,1)$ for $\lambda = 13$.

Lastly, we consider the case for $n < 0$. Up until (11.3.23), we did not use the fact that $|q| < 1$, except that if $|q| > 1$, by Theorem 11.2.1, the left side of (11.3.23) would equal 0. Instead of expanding the summands on the left side of (11.3.23) in powers of q, we expand the summands in powers of q^{-1} when $|q| > 1$. We thus find that, for $|q| > e^{\pi\sqrt{3}/2}$,

$$\begin{aligned} 0 &= \frac{3}{G}\left\{ \sum_{n=1}^{\infty} (-1)^n e^{-n\pi\sqrt{3}} q^{-2n} - \frac{1}{27}\sum_{n=1}^{\infty} (-1)^n e^{-n\pi\sqrt{3}/3} q^{-2n} \right. \\ &\qquad \left. + \sum_{\substack{(\lambda) \\ \lambda>3}} \left(\frac{1}{(c\rho+d)^6} \sum_{n=1}^{\infty} \underline{q}^{2n} q^{-2n} + \frac{1}{(d\rho+c)^6} \sum_{n=1}^{\infty} \underline{\overline{q}}^{2n} q^{-2n} \right) \right\} \\ &= \sum_{n=1}^{\infty} \beta_{-n} q^{-2n}. \end{aligned}$$

This then completes the proof of the entry for $n < 0$.

Thus the proof of Entry 11.3.1 is complete apart from several technical lemmas. □

Lemma 11.3.1. *Given a coprime pair of integers (c,d), we can always choose integers a and b such that $ad - bc = 1$ and*

$$\left| ac + bd - \tfrac{1}{2}(ad+bc) \right| \le \tfrac{1}{2}(c^2 - cd + d^2). \tag{11.3.33}$$

Proof. Let (a_1, b_1) be a solution to $ad - bc = 1$. Then the complete set of solutions is $\{(a_1 + mc, b_1 + md) : m \in \mathbb{Z}\}$. Substituting this expression into the left side of (11.3.33), we see that

$$\begin{aligned}&\left|(a_1+mc)c+(b_1+md)d-\tfrac{1}{2}(a_1+mc)d-\tfrac{1}{2}(b_1+md)c\right|\\&\qquad=\left|a_1c+b_1d-\tfrac{1}{2}a_1d-\tfrac{1}{2}b_1c+m(c^2-cd+d^2)\right|.\end{aligned}$$

For some unique integer m_1, we have

$$a_1c+b_1d-\tfrac{1}{2}a_1d-\tfrac{1}{2}b_1c+m_1(c^2-cd+d^2)\le 0$$

and

$$a_1c+b_1d-\tfrac{1}{2}a_1d-\tfrac{1}{2}b_1c+(m_1+1)(c^2-cd+d^2)\ge 0,$$

since $c^2-cd+d^2>0$. Thus one of the two pairs

$$a=a_1+m_1c,\quad b=b_1+m_1d\qquad\text{and}\qquad a=a_1+(m_1+1)c,\quad b=b_1+(m_1+1)d$$

is our desired solution. □

Lemma 11.3.2. *If $ad-bc=1$, where a, b, c, $d\in\mathbb{Z}$, then the quantity*

$$\underline{q}=\exp\left(\frac{-\pi\sqrt{3}}{2(c^2-cd+d^2)}\right)\exp\left(\pi i\left(\frac{ac+bd-\frac{1}{2}(ad+bc)}{c^2-cd+d^2}\right)\right)\tag{11.3.34}$$

cannot be real.

Proof. By (11.2.34), if c and d are fixed and (a,b) is a solution to $ad-bc=1$ that leads to $\underline{q}$, then other solutions to $ad-bc=1$ will lead to either $\underline{q}$ or $-\underline{q}$. Therefore if $\underline{q}$ is real for some solution (a,b), then it is real for all solutions.

Suppose that a certain pair (c,d) leads to a value of $\underline{q}$ that is real. We can assume without loss of generality that (a,b) satisfies (11.3.33). Since $\underline{q}$ is real, we have, by (11.3.34),

$$ac+bd-\tfrac{1}{2}(ad+bc)\equiv 0\,(\mathrm{mod}\,(c^2-cd+d^2)),$$

and so, by (11.3.33), $ac+bd-\frac{1}{2}(ad+bc)=0$. Adding the equations $0=(ac+bd-\frac{1}{2}(ad+bc))^2$ and $1=(ad-bc)^2$ gives

$$\begin{aligned}1&=a^2c^2+a^2d^2+b^2c^2+b^2d^2-a^2cd-abc^2-abd^2-b^2cd\\&\quad+\tfrac{1}{4}(a^2d^2+2abcd+b^2c^2)\\&=(c^2-cd+d^2)(a^2-ab+b^2)+\tfrac{1}{4}(ad-bc)^2\\&=(c^2-cd+d^2)(a^2-ab+b^2)+\tfrac{1}{4}.\end{aligned}$$

Hence,

$$\tfrac{3}{4}=(c^2-cd+d^2)(a^2-ab+b^2),$$

which is impossible since the variables are integers. Thus $\underline{q}$ cannot be real. □

Lemma 11.3.3. *Under the conditions of Lemma* 11.3.2*, the quantity* $\underline{q}$ *is purely imaginary only when* $\lambda = 1$ *or* $\lambda = 3$*, where* $\lambda = c^2 - cd + d^2$*.*

Proof. Suppose that a certain pair (c, d) leads to a value of $\underline{q}$ that is purely imaginary. We can assume without loss of generality that (a, b) satisfies (11.3.33). Since $\underline{q}$ is purely imaginary, we have, by (11.3.34),

$$2\left|ac + bd - \tfrac{1}{2}(ad + bc)\right| = c^2 - cd + d^2. \tag{11.3.35}$$

Thus, by (11.3.35) and the equality $ad - bc = 1$,

$$\begin{aligned}
&\left(\left|ac + bd - \tfrac{1}{2}(ad + bc)\right| - (a^2 - ab + b^2)\right)^2 \\
&\quad = \left(ac + bd - \tfrac{1}{2}(ad + bc)\right)^2 - 2\left|ac + bd - \tfrac{1}{2}(ad + bc)\right|(a^2 - ab + b^2) \\
&\qquad + (a^2 - ab + b^2)^2 \\
&\quad = \left(ac + bd - \tfrac{1}{2}(ad + bc)\right)^2 - (c^2 - cd + d^2)(a^2 - ab + b^2) + (a^2 - ab + b^2)^2 \\
&\quad = (a^2 - ab + b^2)^2 - \tfrac{3}{4}(ad - bc)^2 \\
&\quad = (a^2 - ab + b^2)^2 - \tfrac{3}{4} \\
&\quad = \Lambda^2 - \tfrac{3}{4}\,,
\end{aligned} \tag{11.3.36}$$

where

$$\Lambda := a^2 - ab + b^2.$$

Since $\Lambda \in \mathbb{Z}$,

$$|ac + bd - \tfrac{1}{2}(ad + bc)| = \tfrac{1}{2}W\,, \tag{11.3.37}$$

where W is odd and positive. Therefore, by (11.3.36) and (11.3.37),

$$\left(\tfrac{1}{2}W - \Lambda\right)^2 = \Lambda^2 - \tfrac{3}{4},$$

or

$$W\left(\tfrac{1}{4}W - \Lambda\right) = -\tfrac{3}{4}\,.$$

Hence, the quantity $\frac{1}{4}W - \Lambda$ is negative. Clearly, its absolute value is at least $\frac{1}{4}$. Since

$$\tfrac{3}{4} = |W\left(\tfrac{1}{4}W - \Lambda\right)| \geq |\tfrac{1}{4}W|,$$

we deduce that $W = 1$ or $W = 3$. But by (11.3.37) and (11.3.35),

$$W = 2|ac + bd - \tfrac{1}{2}(ad + bc)| = c^2 - cd + d^2 = \lambda.$$

We conclude that if $\underline{q}$ is purely imaginary, then λ can be only 1 or 3. When $\lambda = 1$ and $(c, d) = (1, 0)$, say, we have, from (11.3.34),

$$\underline{q} = \pm e^{-\pi\sqrt{3}/2} e^{\pi i/2} = \pm i e^{-\pi\sqrt{3}/2}.$$

When $\lambda = 3$ and $(c, d) = (2, 1)$, say, we have

$$\underline{q} = \pm e^{-\pi/(2\sqrt{3})} e^{-\pi i/2} = \mp i e^{-\pi/(2\sqrt{3})}.$$

So if $\lambda = 1$ or $\lambda = 3$, the quantity $\underline{q}$ is indeed purely imaginary. $\square$

In Lemma 11.3.4 we establish (11.3.19) by proving the contrapositive statement, namely, that if two different solutions (c, d) to the equation $\lambda = c^2 - cd + d^2$ lead to the same set of poles, then the solutions are not distinct.

Lemma 11.3.4. *If* (c_6, d_6) *and* (c_7, d_7) *are two pairs of coprime integers such that*

$$c_6^2 - c_6 d_6 + d_6^2 = c_7^2 - c_7 d_7 + d_7^2 = \lambda, \tag{11.3.38}$$

and if

$$\{\pm \underline{q}_6, \pm \bar{\underline{q}}_6\} = \{\pm \underline{q}_7, \pm \bar{\underline{q}}_7\}, \tag{11.3.39}$$

where $\underline{q}$ *is defined in* (11.3.15)*, then the two solutions of* (11.3.38) *are not distinct. In other words,*

$$\{c_7, d_7\} \cap \{\pm c_6, \pm d_6\} \neq \emptyset. \tag{11.3.40}$$

Proof. From (11.2.34), we see that for fixed (c_j, d_j), the set $\{\pm \underline{q}_j, \pm \bar{\underline{q}}_j\}$ is not affected by our choice of (a_j, b_j). Thus we can assume without loss of generality that (a_j, b_j) satisfies (11.3.33) for $j = 6$ and 7. In other words, if we define e_j by

$$e_j = a_j c_j + b_j d_j - \tfrac{1}{2} a_j d_j - \tfrac{1}{2} b_j c_j, \tag{11.3.41}$$

then we can assume that

$$|e_j| \leq \tfrac{1}{2}\lambda, \quad j = 6, 7. \tag{11.3.42}$$

We show that (11.3.39) implies that $e_6 = \pm e_7$ by considering four different cases.

If $\underline{q}_6 = \underline{q}_7$, then by (11.3.15) and (11.3.41), $\pi i e_6/\lambda = \pi i e_7/\lambda + 2\pi i m$, where m is some integer. In other words,

$$e_6 \equiv e_7 \,(\mathrm{mod}\, 2\lambda). \tag{11.3.43}$$

If, however, $\underline{q}_6 = \bar{\underline{q}}_7$, then, by (11.3.15) and (11.3.41), $\pi i e_6/\lambda = -\pi i e_7/\lambda + 2\pi i m_1$, where m_1 is some integer. Thus,

$$e_6 \equiv -e_7 \,(\mathrm{mod}\, 2\lambda). \tag{11.3.44}$$

If $\underline{q}_6 = -\underline{q}_7$, then, by (11.3.15) and (11.3.41), $\pi i e_6/\lambda = \pi i e_7/\lambda + \pi i + 2\pi i m_2$, where m_2 is some integer, or in other words,

$$e_6 \equiv e_7 + \lambda \,(\mathrm{mod}\, 2\lambda)\,. \tag{11.3.45}$$

Similarly, if $\underline{q}_6 = -\bar{\underline{q}}_7$, then $\pi i e_6/\lambda = -\pi i e_7/\lambda + \pi i + 2\pi i m_3$, where m_3 is some integer, which implies that

$$e_6 \equiv -e_7 + \lambda \,(\mathrm{mod}\, 2\lambda). \tag{11.3.46}$$

From (11.3.43)–(11.3.46), we see that the set equality $\{\pm \underline{q}_6, \pm \underline{\bar{q}}_6\} = \{\pm \underline{q}_7, \pm \underline{\bar{q}}_7\}$ implies that $e_6 \equiv \pm e_7 \pmod{\lambda}$, which implies that, by (11.3.42),

$$e_6 = \pm e_7. \tag{11.3.47}$$

Observe that, by (11.2.4) and (11.3.41),

$$e_j^2 + 1 = e_j^2 + (a_j d_j - b_j c_j)^2 = (c_j^2 - c_j d_j + d_j^2)(a_j^2 - a_j b_j + b_j^2) + \tfrac{1}{4}.$$

Therefore since $e_6^2 = e_7^2$ and $c_6^2 - c_6 d_6 + d_6^2 = c_7^2 - c_7 d_7 + d_7^2$, we deduce that

$$a_6^2 - a_6 b_6 + b_6^2 = a_7^2 - a_7 b_7 + b_7^2. \tag{11.3.48}$$

Later we use this observation.

We now prove (11.3.40) using matrices. We consider two cases.

Case 1. Assume that $e_6 = e_7$. If we let, for $j = 1, 2$,

$$M_j := \begin{bmatrix} c_j - \frac{1}{2} d_j & \frac{\sqrt{3}}{2} d_j \\ & \\ a_j - \frac{1}{2} b_j & \frac{\sqrt{3}}{2} b_j \end{bmatrix}, \tag{11.3.49}$$

then

$$M_j M_j^T = \begin{bmatrix} c_j^2 - c_j d_j + d_j^2 & a_j c_j + b_j d_j - \frac{1}{2} a_j d_j - \frac{1}{2} b_j c_j \\ & \\ a_j c_j + b_j d_j - \frac{1}{2} a_j d_j - \frac{1}{2} b_j c_j & a_j^2 - a_j b_j + b_j^2 \end{bmatrix}. \tag{11.3.50}$$

Observe that

$$M_6 M_6^T = M_7 M_7^T, \tag{11.3.51}$$

by (11.3.38), (11.3.41), (11.3.48), and the assumption that $e_6 = e_7$. After multiplying both sides of (11.3.51) by M_7^{-1} on the left side and then by $(M_6^T)^{-1}$ on the right side, we obtain

$$U := M_7^{-1} M_6 = M_7^T (M_6^T)^{-1} = (M_6^{-1} M_7)^T = (U^{-1})^T. \tag{11.3.52}$$

We want to determine the entries of U, because these may give us information about the entries of the matrices M_6 and M_7. We start by calculating the values of the determinants $|M_j|$ and $|U|$. From the definition of M_j in (11.3.49), a straightforward calculation gives

$$|M_j| = -\frac{\sqrt{3}}{2}, \tag{11.3.53}$$

by (11.2.4). Thus, by (11.3.52) and (11.3.53),

$$|U| = \left(-\frac{2}{\sqrt{3}}\right)\left(-\frac{\sqrt{3}}{2}\right) = 1. \tag{11.3.54}$$

If

$$U = \begin{bmatrix} w & x \\ y & z \end{bmatrix}, \tag{11.3.55}$$

then, by (11.3.54) and (11.3.52), we find that

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} = \begin{bmatrix} z & -y \\ -x & w \end{bmatrix}.$$

Thus, U is of the form

$$U = \begin{bmatrix} w & x \\ -x & w \end{bmatrix}. \tag{11.3.56}$$

By (11.3.52), (11.3.53), and a straightforward calculation (in the notation (11.3.55)),

$$\begin{cases} w = -b_7(c_6 - \frac{1}{2}d_6) + d_7(a_6 - \frac{1}{2}b_6), \\ x = -\frac{\sqrt{3}}{2}b_7d_6 + \frac{\sqrt{3}}{2}b_6d_7, \\ y = \frac{2}{\sqrt{3}}(a_7 - \frac{1}{2}b_7)(c_6 - \frac{1}{2}d_6) - \frac{2}{\sqrt{3}}(a_6 - \frac{1}{2}b_6)(c_7 - \frac{1}{2}d_7), \\ z = d_6(a_7 - \frac{1}{2}b_7) - b_6(c_7 - \frac{1}{2}d_7). \end{cases} \tag{11.3.57}$$

Thus, we see that U has the form

$$U = \begin{bmatrix} \frac{1}{2}\,P & \frac{\sqrt{3}}{2}\,Q \\ \frac{1}{2\sqrt{3}}\,R & \frac{1}{2}\,S \end{bmatrix}, \tag{11.3.58}$$

where P, Q, R, $S \in \mathbb{Z}$. By (11.3.56) and (11.3.58), we conclude that U is of the form

$$U = \begin{bmatrix} \frac{1}{2}\,P & \frac{\sqrt{3}}{2}\,Q \\ -\frac{\sqrt{3}}{2}\,Q & \frac{1}{2}\,P \end{bmatrix}, \tag{11.3.59}$$

where P, $Q \in \mathbb{Z}$.

By (11.3.54) and (11.3.59), we deduce that

$$\frac{1}{4}\,P^2 + \frac{3}{4}\,Q^2 = 1,$$

so either P, $Q \in \{\pm 1\}$ or $P = \pm 2$ and $Q = 0$.

If $P = \pm 2$ and $Q = 0$, then

$$U = \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{11.3.60}$$

But by (11.3.52), we deduce that

$$M_6 = \pm M_7\,, \tag{11.3.61}$$

which implies that $d_6 = \pm d_7$. Thus, the statement we wanted to prove, (11.3.40), holds.

If, however, P, $Q \in \{\pm 1\}$, then U is of the form

$$U = \begin{bmatrix} \pm\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \pm\frac{1}{2} \end{bmatrix} \qquad \text{or} \qquad U = \begin{bmatrix} \pm\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \pm\frac{1}{2} \end{bmatrix}. \tag{11.3.62}$$

We consider two subcases.

Case 1A. Assume the first case in (11.3.62). By the definition of U in (11.3.52),

$$\begin{bmatrix} c_6 - \frac{1}{2}d_6 & \frac{\sqrt{3}}{2}d_6 \\ a_6 - \frac{1}{2}b_6 & \frac{\sqrt{3}}{2}b_6 \end{bmatrix} = \begin{bmatrix} c_7 - \frac{1}{2}d_7 & \frac{\sqrt{3}}{2}d_7 \\ a_7 - \frac{1}{2}b_7 & \frac{\sqrt{3}}{2}b_7 \end{bmatrix} \begin{bmatrix} \pm\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \pm\frac{1}{2} \end{bmatrix}. \tag{11.3.63}$$

The entry in the first row and second column of the matrix on the left-hand side of the equation is

$$\frac{\sqrt{3}}{2}\, d_6 = \frac{\sqrt{3}}{2}\left(c_7 - \frac{1}{2}d_7\right) \pm \frac{\sqrt{3}}{4}d_7 = \frac{\sqrt{3}}{2}c_7 + \left(-\frac{\sqrt{3}}{4} \pm \frac{\sqrt{3}}{4}\right) d_7\,. \tag{11.3.64}$$

If we choose the plus sign in the first matrix of (11.3.62) (and hence in (11.3.64)), then we conclude that $d_6 = c_7$, from which (11.3.40) follows. If, however, we choose the minus sign in (11.3.62), then we conclude that $d_6 = c_7 - d_7$. We show that this implies that $c_6 = c_7$ or $c_6 = -d_7$, from which (11.3.40) follows.

The pairs (c_6, d_6) and (c_7, d_7) are solutions to the equation $\lambda = c^2 - cd + d^2$. Note that if λ and d_6 are fixed, then there are at most two solutions c_6 to the equation. If we set $d_6 = c_7 - d_7$, then it follows from (11.3.2) that two solutions for c_6 (indeed, the only two possible solutions for c_6) are $c_6 = c_7$ and $c_6 = -d_7$. It follows that (11.3.40) holds, and the proof for Case 1A is complete.

Case 1B. The proof for Case 1B is very similar to that for Case 1A. Assume that the second case in (11.3.62) holds. Note that

$$U = \begin{bmatrix} \pm\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \pm\frac{1}{2} \end{bmatrix} = -\begin{bmatrix} \mp\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \mp\frac{1}{2} \end{bmatrix},$$

which is the matrix in Case 1A multiplied by the scalar -1. Thus, by (11.3.63), or (11.3.64),

$$\frac{\sqrt{3}}{2}\, d_6 = -\left(\frac{\sqrt{3}}{2}c_7 + \left(-\frac{\sqrt{3}}{4} \pm \frac{\sqrt{3}}{4}\right) d_7\right), \tag{11.3.65}$$

which implies that $d_6 = -c_7$ or $d_6 = -c_7 + d_7$. But note that $d_6 = d_7 - c_7$ would imply that $c_6 = -c_7$ or $c_6 = d_7$ by (11.3.2). Hence, (11.3.40) follows, and the proof for Case 1B is complete.

Case 2. The proof of Case 2 is very similar to that of Case 1. Assume that

$$e_6 = -e_7. \tag{11.3.66}$$

If we let

$$\tilde{M}_7 = \begin{bmatrix} c_7 - \frac{1}{2}d_7 & -\frac{\sqrt{3}}{2}d_7 \\ -\left(a_7 - \frac{1}{2}b_7\right) & \frac{\sqrt{3}}{2}b_7 \end{bmatrix}, \tag{11.3.67}$$

then a brief calculation gives

$$\tilde{M}_7\tilde{M}_7^T = \begin{bmatrix} c_7^2 - c_7d_7 + d_7^2 & -a_7c_7 - b_7d_7 + \frac{1}{2}a_7d_7 + \frac{1}{2}b_7c_7 \\ -a_7c_7 - b_7d_7 + \frac{1}{2}a_7d_7 + \frac{1}{2}b_7c_7 & a_7^2 - a_7b_7 + b_7^2 \end{bmatrix}.$$

Note that the definition of $\tilde{M}_7$ is the same as that of M_7 in (11.3.49), except that the entries along one diagonal are multiplied by -1. A straightforward calculation gives

$$|\tilde{M}_7| = -\frac{\sqrt{3}}{2}. \tag{11.3.68}$$

By (11.3.50), (11.3.38), (11.3.48), and (11.3.66), $M_6M_6^T = \tilde{M}_7\tilde{M}_7^T$, or

$$\tilde{U} := \tilde{M}_7^{-1}M_6 = (M_6^{-1}\tilde{M}_7)^T. \tag{11.3.69}$$

Then, from the definitions of M_6 in (11.3.49) and $\tilde{M}_7$ in (11.3.67) and the value of $|\tilde{M}_7|$ from (11.3.68) we find that, after multiplying the requisite matrices, if

$$\tilde{U} := \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then

$$\begin{cases} a = -b_7(c_6 - \frac{1}{2}d_6) - d_7(a_6 - \frac{1}{2}b_6), \\ b = -\frac{\sqrt{3}}{2}b_7d_6 - \frac{\sqrt{3}}{2}b_6d_7, \\ c = -\frac{2}{\sqrt{3}}(a_7 - \frac{1}{2}b_7)(c_6 - \frac{1}{2}d_6) - \frac{2}{\sqrt{3}}(a_6 - \frac{1}{2}b_6)(c_7 - \frac{1}{2}d_7), \\ d = -d_6(a_7 - \frac{1}{2}b_7) - b_6(c_7 - \frac{1}{2}d_7). \end{cases}$$

Thus, $\tilde{U}$ has the shape

$$\tilde{U} = \begin{bmatrix} \frac{1}{2}\tilde{P} & \frac{\sqrt{3}}{2}\tilde{Q} \\ \frac{1}{2\sqrt{3}}\tilde{R} & \frac{1}{2}\tilde{S} \end{bmatrix}, \tag{11.3.70}$$

where $\tilde{P}$, $\tilde{Q}$, $\tilde{R}$, $\tilde{S} \in \mathbb{Z}$.

As in Case 1, where (11.3.51) implies (11.3.52) and thus (11.3.53), the condition (11.3.69) implies that $\tilde{U} = (\tilde{U}^{-1})^T$, so that $\tilde{U}$ is of the form

$$\tilde{U} = \begin{bmatrix} \tilde{w} & \tilde{x} \\ -\tilde{x} & \tilde{w} \end{bmatrix}. \tag{11.3.71}$$

By (11.3.70) and (11.3.71), we see that $\tilde{U}$ is of the form

$$\tilde{U} = \begin{bmatrix} \frac{1}{2}\,\tilde{P} & \frac{\sqrt{3}}{2}\,\tilde{Q} \\ -\frac{\sqrt{3}}{2}\,\tilde{Q} & \frac{1}{2}\,\tilde{P} \end{bmatrix}, \tag{11.3.72}$$

where $\tilde{P}$, $\tilde{Q} \in \mathbb{Z}$. But note that, by (11.3.68) and (11.3.53),

$$|\tilde{U}| = |\tilde{M}_7^{-1}||M_6| = \left(-\frac{2}{\sqrt{3}}\right)\left(-\frac{\sqrt{3}}{2}\right) = 1,$$

which implies that

$$\frac{1}{4}\tilde{P}^2 + \frac{3}{4}\tilde{Q}^2 = 1.$$

If $\tilde{P} = \pm 2$ and $\tilde{Q} = 0$, then, by (11.3.60) and (11.3.61), $d_6 = \pm d_7$. If, however, $\tilde{P}$, $\tilde{Q} \in \{\pm 1\}$, then $\tilde{U}$ has the form

$$\tilde{U} = \begin{bmatrix} \pm\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \pm\frac{1}{2} \end{bmatrix} \qquad \text{or} \qquad \tilde{U} = \begin{bmatrix} \pm\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \pm\frac{1}{2} \end{bmatrix}. \tag{11.3.73}$$

If the first case of (11.3.73) holds, then, by (11.3.69),

$$\begin{bmatrix} c_6 - \frac{1}{2}d_6 & \frac{\sqrt{3}}{2}d_6 \\ a_6 - \frac{1}{2}b_6 & \frac{\sqrt{3}}{2}b_6 \end{bmatrix} = \begin{bmatrix} c_7 - \frac{1}{2}d_7 & -\frac{\sqrt{3}}{2}d_7 \\ -\left(a_7 - \frac{1}{2}b_7\right) & \frac{\sqrt{3}}{2}b_7 \end{bmatrix} \begin{bmatrix} \pm\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \pm\frac{1}{2} \end{bmatrix}. \tag{11.3.74}$$

The entry in the first row and second column of the matrix on the left-hand side of the equation is

$$\frac{\sqrt{3}}{2}d_6 = \frac{\sqrt{3}}{2}\left(c_7 - \frac{1}{2}d_7\right) \mp \frac{\sqrt{3}}{4}d_7 = \frac{\sqrt{3}}{2}c_7 + \left(-\frac{\sqrt{3}}{4} \mp \frac{\sqrt{3}}{4}\right)d_7, \tag{11.3.75}$$

which implies (11.3.40), because (11.3.75) is identical to equation (11.3.64), which ultimately implied (11.3.40).

If, however, the second option in (11.3.73) holds, then by a similar argument,

$$\frac{\sqrt{3}}{2}d_6 = -\frac{\sqrt{3}}{2}\left(c_7 - \frac{1}{2}d_7\right) \mp \frac{\sqrt{3}}{4}d_7 = -\frac{\sqrt{3}}{2}c_7 + \left(\frac{\sqrt{3}}{4} \mp \frac{\sqrt{3}}{4}\right)d_7, \tag{11.3.76}$$

which implies (11.3.40), because (11.3.76) is identical to equation (11.3.65), which ultimately implied (11.3.40).

Thus, (11.3.40) holds in both Case 1 and Case 2, and the lemma is proved. □

11.4 The Coefficients of $Q(q)/R(q)$

In a letter to Hardy written from Matlock House, an English sanitarium, Ramanujan [244, p. 117] communicated a result that is very similar to formula (11.3.6) in Entry 11.3.1. We prove the result in this section.

Suppose that (c,d) is a pair of coprime integers that is a solution to the equation

$$\mu = c^2 + d^2, \tag{11.4.1}$$

where μ is fixed. Let $K = \mathbb{Q}(\sqrt{-1})$. Then $\mathfrak{O} = \mathbb{Z}[\sqrt{-1}] = \mathbb{Z} \oplus \mathbb{Z}i$ is a principal ideal domain. Thus, from (11.4.1), $\lambda = N(\mathfrak{A}) = \mathfrak{A}\bar{\mathfrak{A}}$, where $\mathfrak{A} = (c+di) =: (\alpha)$ and $\bar{\mathfrak{A}} = (\bar{\alpha})$. The group of units in $\mathfrak{O}_K$ is then given by $U = \{\pm 1, \pm i\}$. Thus,

$$\begin{aligned}\mathfrak{A} &= (\alpha) = (-\alpha) = (i\alpha) = (-i\alpha),\\ \bar{\mathfrak{A}} &= (\bar{\alpha}) = (-\bar{\alpha}) = (i\bar{\alpha}) = (-i\bar{\alpha}).\end{aligned}$$

Hence one solution generates a total of eight solutions, namely,

$$\pm(c,d), \quad \pm(c,-d), \quad \pm(d,c), \quad \pm(d,-c). \tag{11.4.2}$$

Definition 11.4.1. *We say that two solutions* (c_1,d_1) *and* (c_2,d_2) *to the equation* (11.4.1) *are distinct if they do not simultaneously belong to the same set of solutions in* (11.4.2).

Note that (c_1,d_1) and (c_2,d_2) are distinct solutions to (11.4.1) if and only if

$$c_2 \notin \{\pm c_1, \pm d_1\}. \tag{11.4.3}$$

Recall that [223, p. 164] the integers μ that can be represented in the form $\mu = c^2 + d^2$, with c and d coprime, are integers of the form

$$\mu = 2^a \prod_{j=1}^{r} p_j^{a_j}, \tag{11.4.4}$$

where $a = 0$ or 1, p_j is a prime of the form $4m+1$, and a_j is a nonnegative integer, $1 \le j \le r$.

Entry 11.4.1 (p. 117). *Let*

$$\frac{Q(q^2)}{R(q^2)} = \sum_{n=0}^{\infty} \delta_n q^{2n}, \qquad |q| < q_0, \tag{11.4.5}$$

and

$$J = Q(e^{-2\pi}) = 1 + 240 \sum_{k=1}^{\infty} \frac{k^3}{e^{2\pi k} - 1} = 1.45576\ldots. \tag{11.4.6}$$

Then, if $n \ge 0$,

$$\delta_n = \frac{2}{J}\left\{ e^{2n\pi} - \frac{(-1)^n}{2^2}e^{n\pi} + \frac{2\cos\left(\dfrac{4\pi n}{5} + 4\arctan 2\right)}{5^2}e^{2n\pi/5} \right.$$
$$\left. + \frac{2\cos\left(\dfrac{3\pi n}{5} + 4\arctan 3\right)}{10^2}e^{2n\pi/10} + \cdots \right\}$$
$$:= \frac{2}{J}\sum_{(\mu)} \frac{v_\mu(n)}{\mu^2}e^{2n\pi/\mu}. \tag{11.4.7}$$

Here, μ runs over the integers of the form (11.4.4),

$$v_1(n) = 1, \qquad v_2(n) = (-1)^{n+1}, \tag{11.4.8}$$

and, for $\mu \geq 5$,

$$v_\mu(n) = 2\sum_{c,d} \cos\left((ac+bd)\frac{2n\pi}{\mu} + 4\arctan\frac{c}{d}\right), \tag{11.4.9}$$

where the sum is over all pairs (c, d), where (c, d) is a distinct solution to $\mu = c^2 + d^2$ and (a, b) is any solution to $ad - bc = 1$. Also, distinct solutions (c, d) to $\mu = c^2 + d^2$ give rise to distinct terms in the sum in (11.4.7). *If $n < 0$, then the sum on the far right side of* (11.4.7) *equals* 0.

Proof. Let $|q| < 1$. We apply Theorem 11.2.1 to the function $f(q) = Q(q^2)/R(q^2)$. Then $\varphi(\tau) = E_4(\tau)/E_6(\tau)$. Since $E_4(\tau)$ and $E_6(\tau)$ are modular forms of degrees -4 and -6, respectively, $\varphi(\tau)$ satisfies the functional equation (11.2.1) with $n = 2$. The only zero of $E_6(\tau)$ in P_1 is at

$$\tau = i \tag{11.4.10}$$

(Rankin [246, p. 198]), while by (11.3.10), $E_4(\tau)$ does not have a zero at $\tau = i$. Thus, in the notation of Theorem 11.2.1,

$$\alpha = i. \tag{11.4.11}$$

Lastly, because $1/E_6(\tau)$ is meromorphic in the upper half-plane [255, p. 50], we see that $\varphi(\tau)$ is also meromorphic there and that $f(q)$ is meromorphic inside the unit circle.

We now calculate $A = \mathrm{Res}(\varphi, i)$. By (11.1.9), (11.3.11), and (11.4.6),

$$\mathrm{Res}(f, e^{-\pi}) = \left.\frac{Q(q^2)}{dR(q^2)/dq}\right|_{q=e^{-\pi}} = \left.\frac{-qQ(q^2)}{Q^2(q^2) - P(q^2)R(q^2)}\right|_{q=e^{-\pi}}$$
$$= \left.\frac{-qQ(q^2)}{Q^2(q^2)}\right|_{q=e^{-\pi}} = \frac{-e^{-\pi}}{J}. \tag{11.4.12}$$

Thus, by (11.4.12) and (11.2.37),

$$A = \operatorname{Res}(\varphi, i) = \frac{\operatorname{Res}(f, e^{-\pi})}{\pi i e^{-\pi}} = \frac{-1}{J\pi i}\,. \tag{11.4.13}$$

By (11.2.2), (11.4.11), and (11.4.13), we deduce that

$$f(q) = \frac{2}{J}\sum_{(c,d)} \frac{1}{(ci+d)^4}\frac{1}{1-(q/\underline{q})^2}\,, \tag{11.4.14}$$

where

$$\underline{q} = \exp\left(\pi i\left(\frac{ai+b}{ci+d}\right)\right), \tag{11.4.15}$$

and the conditions on a, b, c, and d are the same as in (11.3.13) and (11.3.14). We need to explicitly determine the values over which (c,d) runs.

The analysis used to determine which pairs (c,d) are counted in the summation is very similar to that in Entry 11.3.1. Now,

$$\underline{q} = \exp\left(\pi i\left(\frac{(ac+bd)+i}{c^2+d^2}\right)\right) = \exp\left(\frac{-\pi}{\mu}\right)\exp\left(\frac{\pi i}{\mu}(ac+bd)\right), \tag{11.4.16}$$

where $\mu = c^2+d^2$. As in (11.3.16), we can show that if the two pairs (c_1,d_1) and (c_2,d_2) produce distinct values of μ, then they lead to distinct values for the set $\{\pm\underline{q}\}$, i.e.,

$$\{\pm\underline{q}_1\}\cap\{\pm\underline{q}_2\} = \emptyset. \tag{11.4.17}$$

When $\mu = 1$ or $\mu = 2$, then only four of the eight pairs in (11.4.2) are distinct in this case. Each value of μ corresponds to only two values of $\underline{q}$, say, $\{\pm\underline{q}_3\}$. When $\mu \geq 5$, all eight pairs in (11.4.2) are distinct. However, if (c,d) leads to $\{\pm\underline{q}_4\}$, say, then $(-c,d)$ and (d,c) each lead to $\{\pm\bar{\underline{q}}_4\}$. It follows that the eight pairs of solutions in (11.4.2) lead to only four different poles, namely,

$$\{\pm q_4, \pm\bar{\underline{q}}_4\}\,. \tag{11.4.18}$$

These four poles are indeed distinct, because as we show in Lemmas 11.4.1–11.4.3 at the end of the section, $\underline{q}$ is real only for $\mu = 1$, and $\underline{q}$ is purely imaginary only for $\mu = 2$.

Lastly, in Lemma 11.4.4, we prove that if $\mu \geq 5$ and (c_1,d_1) and (c_2,d_2), say, are distinct solutions to equation (11.4.1), then each solution, taken together with the seven corresponding solutions in (11.4.2), yields four distinct poles, i.e.,

$$\{\pm\underline{q}_1, \pm\bar{\underline{q}}_1\}\cap\{\pm\underline{q}_2, \pm\bar{\underline{q}}_2\} = \emptyset\,. \tag{11.4.19}$$

In summary, we have so far shown, by (11.4.14) and (11.4.17)–(11.4.19), that

$$f(q) = \frac{2}{J}\left\{\sum_{\substack{(\mu)\\ \mu\leq 2}} \frac{1}{(ci+d)^4}\frac{1}{1-(q/\underline{q})^2}\right. \tag{11.4.20}$$

$$+\sum_{\substack{(\mu)\\ \mu>2}}\left(\frac{1}{(ci+d)^4}\frac{1}{1-(q/\underline{q})^2}+\frac{1}{(-ci+d)^4}\frac{1}{1-(q/\underline{\bar{q}})^2}\right)\Bigg\},$$

where μ runs over all integers of the form (11.4.4), and where, for each fixed μ, the sum is also over all distinct pairs (c,d) satisfying (11.4.1).

For $\mu=1$ and, say, $(a,b,c,d)=(1,0,0,1)$, we find that, by (11.4.16), $\underline{q}=e^{-\pi}$, so that the summand in (11.4.20) is

$$\frac{1}{1-q^2e^{2\pi}}\,. \tag{11.4.21}$$

For $\mu=2$ and, say, $(a,b,c,d)=(1,0,1,1)$, we find that $\underline{q}=ie^{-\pi/2}$, so that the summand in (11.4.20) is

$$-\frac{1}{2^2}\frac{1}{1+q^2e^{\pi}}\,. \tag{11.4.22}$$

Thus, by (11.4.21) and (11.4.22), we can rewrite (11.4.20) as

$$\begin{aligned} f(q)&=\frac{2}{J}\Bigg\{\frac{1}{1-q^2e^{2\pi}}-\frac{1}{2^2}\frac{1}{1+q^2e^{\pi}}\\ &\qquad+\sum_{\substack{(\mu)\\ \mu>2}}\left(\frac{1}{(ci+d)^4}\frac{1}{1-(q/\underline{q})^2}+\frac{1}{(-ci+d)^4}\frac{1}{1-(q/\underline{\bar{q}})^2}\right)\Bigg\}\\ &=\frac{2}{J}\Bigg\{\sum_{n=0}^{\infty}e^{2n\pi}q^{2n}-\frac{1}{2^2}\sum_{n=0}^{\infty}(-1)^ne^{n\pi}q^{2n}\\ &\qquad+\sum_{\substack{(\mu)\\ \mu>2}}\left(\frac{1}{(ci+d)^4}\sum_{n=0}^{\infty}\underline{q}^{-2n}q^{2n}+\frac{1}{(-ci+d)^4}\sum_{n=0}^{\infty}\underline{\bar{q}}^{-2n}q^{2n}\right)\Bigg\}\\ &=\sum_{n=0}^{\infty}\delta_nq^{2n}, \end{aligned} \tag{11.4.23}$$

where $|q|<e^{-\pi}$ and

$$\begin{aligned} \delta_n&=\frac{2}{J}\left\{e^{2n\pi}-\frac{(-1)^n}{2^2}e^{n\pi}+\sum_{\substack{(\mu)\\ \mu>2}}\left(\frac{1}{(ci+d)^4}\underline{q}^{-2n}+\frac{1}{(-ci+d)^4}\underline{\bar{q}}^{-2n}\right)\right\}\\ &=\frac{2}{J}\left\{e^{2n\pi}-\frac{(-1)^n}{2^2}e^{n\pi}+\sum_{\substack{(\mu)\\ \mu>2}}\frac{2\cos\left((ac+bd)\dfrac{2n\pi}{\mu}+4\arctan\dfrac{c}{d}\right)}{\mu^2}e^{2n\pi/\mu}\right\} \end{aligned}$$

$$= \frac{2}{J}\sum_{(\mu)} \frac{v_\mu(n)e^{2n\pi/\mu}}{\mu^2}, \tag{11.4.24}$$

where $v_\mu(n)$ is defined in (11.4.8) and (11.4.9). To obtain the displayed terms in (11.4.7), we choose $(a,b,c,d) = (1,0,2,1)$ for $\mu = 5$ and $(a,b,c,d) = (1,0,3,1)$ for $\mu = 10$.

Thus, apart from the lemmas below, the proof of Entry 11.4.1 is complete for $n \geq 0$. For $n < 0$, we repeat the argument above but with $|q| > 1$. Then, by Theorem 11.2.1, the left side of (11.4.23) equals 0 instead of $f(q)$. We now expand the series on the left side of (11.4.23) in powers of $1/q$ instead of powers of q. We complete the argument as in the proof of Entry 11.3.1. □

The following four lemmas are analogous to Lemmas 11.3.1–11.3.4, and their proofs are very similar. Lemmas 11.4.1–11.4.3 show that the poles of (11.4.18) are distinct. Lemma 11.4.4 is used in the proof of (11.4.19).

Lemma 11.4.1. *Given a pair of coprime integers (c,d), we can always choose integers a and b such that $ad - bc = 1$ and*

$$|ac + bd| \leq \tfrac{1}{2}(c^2 + d^2). \tag{11.4.25}$$

Proof. The proof is virtually identical to that of Lemma 11.3.1. □

Lemma 11.4.2. *If $ad - bc = 1$, where a, b, c, $d \in \mathbb{Z}$, then the quantity*

$$\underline{q} = \exp\left(\frac{-\pi}{c^2+d^2}\right)\exp\left(\frac{\pi i}{c^2+d^2}(ac+bd)\right) \tag{11.4.26}$$

is real only when $c^2 + d^2 = 1$.

Proof. Suppose that a certain pair (c,d) leads to a value of $\underline{q}$ that is real. As in the proof of Lemma 11.3.2, we can assume without loss of generality that (a,b) satisfies (11.4.25).

Since $\underline{q}$ is real, we see that, by (11.4.26) and (11.4.25), $ac+bd = 0$. Adding the equations $0 = (ac+bd)^2$ and $1 = (ad-bc)^2$ gives

$$1 = (ac)^2 + (bd)^2 + (ad)^2 + (bc)^2 = (a^2+b^2)(c^2+d^2), \tag{11.4.27}$$

which implies that $c^2 + d^2 = 1$. □

Lemma 11.4.3. *The quantity $\underline{q}$ is imaginary only when $c^2 + d^2 = 2$.*

Proof. Suppose that a certain pair (c,d) leads to a value of $\underline{q}$ that is imaginary. We can assume without loss of generality that (a,b) satisfies (11.4.25).

Since $\underline{q}$ is purely imaginary, we find that, by (11.4.26) and (11.4.25),

$$2|ac + bd| = c^2 + d^2, \tag{11.4.28}$$

and so

$$\begin{aligned}\big(|ac+bd|-(a^2+b^2)\big)^2 &= (ac+bd)^2-(c^2+d^2)(a^2+b^2)+(a^2+b^2)^2\\ &= -a^2d^2-b^2c^2+2abcd+(a^2+b^2)^2\\ &= -(ad-bc)^2+(a^2+b^2)^2\\ &= -1+(a^2+b^2)^2.\end{aligned}$$

The only squares that differ by 1 are 0 and 1. Thus, $|ac+bd|-(a^2+b^2)=0$ and $a^2+b^2=1$, so that $|ac+bd|=1$. By (11.4.28), this implies that $c^2+d^2=2$. □

In Lemma 11.4.4 we establish (11.4.19) by proving that if two different solutions (c,d) to the equation $\mu=c^2+d^2$ lead to the same set of poles, then the solutions are not distinct.

Lemma 11.4.4. *If* (c_1,d_1) *and* (c_2,d_2), *say, are two pairs of coprime integers such that*

$$c_1^2+d_1^2=c_2^2+d_2^2=\mu, \tag{11.4.29}$$

and if

$$\{\pm \underline{q}_1, \pm \underline{\bar{q}}_1\}=\{\pm \underline{q}_2, \pm \underline{\bar{q}}_2\}, \tag{11.4.30}$$

where $\underline{q}$ *is defined in* (11.4.16), *then the two solutions of* (11.4.29) *are not distinct. In other words,*

$$c_2\in\{\pm c_1,\pm d_1\}. \tag{11.4.31}$$

Proof. As in the proof of Lemma 11.3.4, we can assume that (a_1,b_1) and (a_2,b_2) satisfy (11.4.25), so that

$$-\tfrac12\mu\le g_j:=a_jc_j+b_jd_j\le\tfrac12\mu,\qquad j=1,2. \tag{11.4.32}$$

As in Lemma 11.3.4, (11.4.30) and (11.4.32) imply that

$$g_1=\pm g_2. \tag{11.4.33}$$

Since $ad-bc=1$, by (11.4.32), we find that, for $j=1,2$,

$$g_j^2+1=g_j^2+(a_jd_j-b_jc_j)^2=(a_j^2+b_j^2)(c_j^2+d_j^2), \tag{11.4.34}$$

so that by (11.4.33) and (11.4.29), we deduce that

$$a_1^2+b_1^2=a_2^2+b_2^2. \tag{11.4.35}$$

We now prove (11.4.31) by considering two cases.

Case 1. $g_1=g_2$. If we let

$$L_j=\begin{bmatrix}a_j & b_j\\ c_j & d_j\end{bmatrix},$$

then a simple calculation shows that

$$L_1 L_1^T = L_2 L_2^T,$$

by (11.4.29), (11.4.35), and the assumption that $g_1 = g_2$. This implies that

$$V := L_2^{-1} L_1 = L_2^T (L_1^T)^{-1} = (V^{-1})^T. \tag{11.4.36}$$

Therefore, as in (11.3.56),

$$V = \begin{bmatrix} w & x \\ -x & w \end{bmatrix}, \tag{11.4.37}$$

for some integers w, x. Since $ad - bc = 1$, we know that $|L_j| = 1$, and so, by (11.4.36),

$$|V| = 1. \tag{11.4.38}$$

Clearly, V has integral entries, so that (11.4.37) and (11.4.38) imply that

$$V = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \qquad \text{or} \qquad V = \begin{bmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{bmatrix}. \tag{11.4.39}$$

By (11.4.36) and (11.4.39),

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} \pm a_2 & \pm b_2 \\ \pm c_2 & \pm d_2 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} \mp b_2 & \pm a_2 \\ \mp d_2 & \pm c_2 \end{bmatrix},$$

either of which implies (11.4.31).

Case 2. $g_1 = -g_2$. This case is very similar to that above. We define

$$\tilde{L}_2 = \begin{bmatrix} a_2 & -b_2 \\ -c_2 & d_2 \end{bmatrix}.$$

Then, by a brief calculation,

$$L_1 L_1^T = \tilde{L}_2 \tilde{L}_2^T.$$

If we define $\tilde{V}$ by

$$\tilde{V} = \tilde{L}_2^{-1} L_1, \tag{11.4.40}$$

then, as in Case 1,

$$\tilde{V} = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix} \qquad \text{or} \qquad \tilde{V} = \begin{bmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{bmatrix}. \tag{11.4.41}$$

Because $L_1 = \tilde{L}_2 \tilde{V}$, (11.4.41) implies (11.4.31).

Thus, (11.4.31) holds in Case 1 and Case 2, and the lemma is proved. □

11.5 The Coefficients of $(\pi P(q)/3)/R(q)$ and $(\pi P(q)/3)^2/R(q)$

The following theorem is from the fragment published with the "lost notebook" [244, pp. 102–104] and is similar to the previous theorems. However, since this result involves a function that is not a modular form, we need to modify Theorem 11.2.1 in order to prove the result.

Entry 11.5.1 (p. 102). *Let*

$$\frac{\frac{\pi}{3}P(q^2)}{R(q^2)} = \sum_{n=0}^{\infty} \eta_{1,n} q^{2n}, \qquad |q| < q_0, \tag{11.5.1}$$

and

$$\frac{\left(\frac{\pi}{3}P(q^2)\right)^2}{R(q^2)} = \sum_{n=0}^{\infty} \eta_{2,n} q^{2n}, \qquad |q| < q_0. \tag{11.5.2}$$

If

$$C := 1 + 480 \sum_{k=1}^{\infty} \frac{k^7}{e^{2\pi k} - 1}, \tag{11.5.3}$$

then, if $n \geq 0$,

$$\begin{aligned}\eta_{1,n} &= \frac{2}{C}\left\{ e^{2n\pi} - \frac{(-1)^n}{2^3} e^{n\pi} + \frac{2\cos\left(\frac{4\pi n}{5} + 8\arctan 2\right)}{5^3} e^{2n\pi/5} \right. \\ &\qquad \left. + \frac{2\cos\left(\frac{3\pi n}{5} + 8\arctan 3\right)}{10^3} e^{2n\pi/10} + \cdots \right\} \\ &=: \frac{2}{C} \sum_{(\mu)} \frac{W_\mu(n)}{\mu^3} e^{2n\pi/\mu}\end{aligned} \tag{11.5.4}$$

and

$$\eta_{2,n} = \frac{2}{C} \sum_{(\mu)} \frac{W_\mu(n)}{\mu^2} e^{2n\pi/\mu}, \tag{11.5.5}$$

where μ *runs over the integers of the form* (11.4.4). *Here,*

$$W_1(n) = 1, \qquad W_2(n) = -(-1)^n, \tag{11.5.6}$$

and, for $\mu \geq 5$,

$$W_\mu(n) = 2\sum_{c,d} \cos\left((ac+bd)\frac{2n\pi}{\mu} + 8\arctan\frac{c}{d}\right), \tag{11.5.7}$$

where the sum is over all pairs (c,d), *where* (c,d) *is a distinct solution of* $\mu = c^2 + d^2$. *Also, distinct solutions* (c,d) *to* $\mu = c^2 + d^2$ *give rise to distinct terms in the sums in* (11.5.4) *and* (11.5.5). *If* $n < 0$, *then the sums on the right sides of* (11.5.4) *and* (11.5.5) *are both equal to* 0.

Note that the definition of $W_\mu(n)$ is almost identical to that of $v_\mu(n)$ in Entry 11.4.1. Recall that the definition of *distinct* solutions is given in Definition 11.4.1 and that a and b are any integral solutions of $ad - bc = 1$.

Proof. For $j = 1$ or 2, let $f_j(q)$ denote the quotients on the left sides of (11.5.1) and (11.5.2), respectively. Define $\varphi_j(\tau) = f_j(e^{\pi i\tau})$. Then, by (11.1.5) and (11.1.4),

$$\varphi_1(\tau) = \frac{\pi E_2^*(\tau)/3}{E_6(\tau)} \qquad \text{and} \qquad \varphi_2(\tau) = \frac{(\pi E_2^*(\tau)/3)^2}{E_6(\tau)}. \tag{11.5.8}$$

Recall that $E_2(\tau)$, defined by (11.1.6), satisfies the functional equation

$$E_2(V\tau) = E_2(\tau)(c\tau+d)^2 \tag{11.5.9}$$

for any modular transformation $V\tau = (a\tau+b)/(c\tau+d)$. Although $E_2^*(\tau)$ is not a modular form, we see from Ramanujan's work [53, p. 320] that it does satisfy a modified functional equation:

$$E_2^*(V\tau) = E_2^*(\tau)(c\tau+d)^2 - \frac{6ci}{\pi}(c\tau+d). \tag{11.5.10}$$

Since E_6 is a modular form of weight 6,

$$E_6(V\tau) = E_6(\tau)(c\tau+d)^6. \tag{11.5.11}$$

Taking (11.5.10) and (11.5.11) together, we find that

$$\varphi_1(\tau) = \varphi_1(V\tau)(c\tau+d)^4 + \frac{2ci}{E_6(V\tau)}(c\tau+d)^5 \tag{11.5.12}$$

and, after squaring both sides of (11.5.10),

$$\varphi_2(\tau) = \varphi_2(V\tau)(c\tau+d)^2 + 4ci\varphi_1(V\tau)(c\tau+d)^3 - \frac{4c^2}{E_6(V\tau)}(c\tau+d)^4. \tag{11.5.13}$$

We now prove that modified versions of Theorem 11.2.1 hold when the functional equation (11.2.1) is replaced by either (11.5.12) or (11.5.13).

When we replace (11.2.1) by either (11.5.12) or (11.5.13), the parts of the proof that are affected are the estimation of the integral (11.2.5) and the calculation of the residues of φ_j. The only resulting change in the statement of the theorem itself is a slight modification of (11.2.2).

When we replace (11.2.1) by (11.5.12) or (11.5.13), the function $\varphi_j(\tau)$ satisfies the conditions of Theorem 11.2.1 for $j = 1$ or 2 with $\alpha = i$, because $P(q)$ is analytic in the unit circle.

We now estimate the integral (11.2.5). Using the notation in the paragraph surrounding (11.2.11), we let

$$V^*\tau := (k^{'}\tau - h^{'})/(-k\tau + h). \tag{11.5.14}$$

By (11.5.12) and (11.5.13), our estimates for φ in (11.2.24) are replaced by, respectively,

$$|\varphi_1(\tau)| \le |\varphi_1(V^*\tau)|\,|-k\tau+h|^4 + \frac{2k}{|E_6(V^*\tau)|}|-k\tau+h|^5 \tag{11.5.15}$$

and

$$\begin{aligned}|\varphi_2(\tau)| \le & |\varphi_2(V^*\tau)|\,|-k\tau+h|^2 + 4k\,|\varphi_1(V^*\tau)|\,|-k\tau+h|^3 \\ & + \frac{4k^2}{|E_6(V^*\tau)|}|-k\tau+h|^4 .\end{aligned} \tag{11.5.16}$$

As we have seen, Theorem 11.2.1 can be applied to the function $1/E_6(\tau)$ (as Hardy and Ramanujan [177] did). Thus, on the boundary of P,

$$\left|\frac{1}{E_6(\tau)}\right| < \tilde{M}, \tag{11.5.17}$$

for some positive constant $\tilde{M}$. By (11.5.15), (11.5.17), (11.2.22), and (11.2.25),

$$\begin{aligned}|\varphi_1(\tau)| &< M|-k\tau+h|^4 + 2k\tilde{M}|-k\tau+h|^5 \\ &= k^4M\,|\tau-h/k|^4 + 2k^6\tilde{M}\,|\tau-h/k|^5 \\ &< \frac{16M}{(k^2+kk'+k'^2)^2} + \frac{64k\tilde{M}}{(k^2+kk'+k'^2)^{5/2}} \\ &< \frac{M_0}{(k^2+kk'+k'^2)^2},\end{aligned} \tag{11.5.18}$$

where M_0 is some positive constant.

Similarly,

$$|\varphi_2(\tau)| < \frac{\hat{M}_0}{k^2+kk'+k'^2}, \tag{11.5.19}$$

where $\hat{M}_0$ is some positive constant. Note that the inequalities (11.5.18) and (11.5.19) are similar to (11.2.26). The remainder of the argument is the same as before, and so the integral in (11.2.5) approaches 0 as m approaches ∞ in the cases of φ_1 and φ_2.

Next we evaluate $\operatorname{Res}(\varphi_j, i)$ using the following lemma.

Lemma 11.5.1. *Let $V\tau = (a\tau + b)/(c\tau + d)$, where a, b, c, and d are integers satisfying $ad - bc = 1$. Then*

$$\frac{\pi}{3}E_2^*(Vi) = \frac{\pi}{3}P(\underline{q}^2) = c^2 + d^2, \tag{11.5.20}$$

where

$$\underline{q} = \exp\left(\pi i \left(\frac{ai+b}{ci+d}\right)\right).$$

Proof. Consider the function $E_2(\tau)$ defined in (11.1.6). From [51, p. 159], [53, p. 256],

$$E_2(i) = 1 - 24\sum_{k=1}^{\infty}\frac{k}{e^{2\pi k}-1} - \frac{3}{\pi} = 0. \tag{11.5.21}$$

By (11.5.21), (11.1.6), and the definition of $\underline{q}$ above,

$$0 = E_2(Vi) = 1 - 24\sum_{k=1}^{\infty}\frac{k\underline{q}^{2k}}{1-\underline{q}^{2k}} - \frac{3}{\pi \mathrm{Im}\,(Vi)} = 1 - 24\sum_{k=1}^{\infty}\frac{k\underline{q}^{2k}}{1-\underline{q}^{2k}} - \frac{3}{\pi}(c^2+d^2),$$

and hence we obtain (11.5.20). □

By (11.2.36), or more precisely the sentence preceding (11.2.36), and (11.4.13),

$$\mathrm{Res}\left(\frac{1}{E_6(\tau)}, Vi\right) = \frac{\mathrm{Res}\,(1/E_6(\tau), i)}{(ci+d)^8} = \frac{-1/(J^2\pi i)}{(ci+d)^8} = \frac{-1/(C\pi i)}{(ci+d)^8},$$

by a result of Ramanujan [240, Table I, Entry 4], [242, p. 141], where J is defined in (11.4.6) and C is defined in (11.5.3). By (11.2.37) and the calculation above,

$$\mathrm{Res}\left(\frac{1}{R(q^2)}, \underline{q}\right) = -\frac{\underline{q}}{C(ci+d)^8}. \tag{11.5.22}$$

It follows from (11.5.20) and (11.5.22) that

$$\mathrm{Res}\left(f_j(q), \underline{q}\right) = -(c^2+d^2)^j\frac{\underline{q}}{C(ci+d)^8}. \tag{11.5.23}$$

When we replace (11.2.37) with (11.5.23), the analogue of (11.2.38) becomes, or, alternatively, (11.2.33) becomes,

$$f_j(q) = \frac{2}{C}\sum_{(c,d)}\frac{\mu^j}{(ci+d)^8}\frac{1}{1-(q/\underline{q})^2}, \tag{11.5.24}$$

where $\mu = c^2 + d^2$.

Thus, the conclusion of Theorem 11.2.1 is valid for φ_j if we replace (11.2.2) by (11.5.24).

Since the quantity $\underline{q}$ is the same for φ_j as for φ in Entry 11.4.1, the analysis involving the values of (c,d) counted in the summation in (11.4.14) is valid for φ_j as well. So by (11.4.20), (11.4.23), and (11.4.24), we find that

$$\begin{aligned} f_j(q) &= \frac{2}{C}\left\{\sum_{\substack{(\mu)\\ \mu\le 2}} \frac{\mu^j}{(ci+d)^8}\frac{1}{1-(q/\underline{q})^2}\right.\\ &\qquad \left. + \sum_{\substack{(\mu)\\ \mu>2}}\left(\frac{\mu^j}{(ci+d)^8}\frac{1}{1-(q/\underline{q})^2} + \frac{\mu^j}{(-ci+d)^8}\frac{1}{1-(q/\overline{\underline{q}})^2}\right)\right\}\\ &= \frac{2}{C}\left\{\frac{1}{1-q^2e^{2\pi}} - \frac{2^j}{2^4}\frac{1}{1+q^2e^{\pi}}\right.\\ &\qquad \left. + \sum_{\substack{(\mu)\\ \mu>2}}\left(\frac{\mu^j}{(ci+d)^8}\frac{1}{1-(q/\underline{q})^2} + \frac{\mu^j}{(-ci+d)^8}\frac{1}{1-(q/\overline{\underline{q}})^2}\right)\right\}\\ &= \sum_{n=0}^{\infty}\eta_{j,n}q^{2n}, \end{aligned}$$

where, for $j = 1, 2$,

$$\begin{aligned} \eta_{j,n} &= \frac{2}{C}\left\{e^{2n\pi} - \frac{(-1)^n}{2^{4-j}}e^{n\pi} + \sum_{\substack{(\mu)\\ \mu>2}}\left(\frac{\mu^j}{(ci+d)^8}\underline{q}^{-2n} + \frac{\mu^j}{(-ci+d)^8}\overline{\underline{q}}^{-2n}\right)\right\}\\ &= \frac{2}{C}\left\{e^{2n\pi} - \frac{(-1)^n}{2^{4-j}}e^{n\pi} + \sum_{\substack{(\mu)\\ \mu>2}}\frac{2\cos\left((ac+bd)\dfrac{2n\pi}{\mu} + 8\arctan\dfrac{d}{c}\right)}{\mu^{4-j}}e^{2n\pi/\mu}\right\}\\ &= \frac{2}{C}\sum_{(\mu)}\frac{W_\mu(n)e^{2n\pi/\mu}}{\mu^{4-j}}, \end{aligned} \tag{11.5.25}$$

by (11.5.7). In (11.5.25), as in (11.4.20), μ runs over all integers of the form (11.4.4), and for each fixed μ, the sum is also over all distinct pairs (c, d).

Thus, the proof of (11.5.4) is complete. □

11.6 The Coefficients of $(\pi P(q)/2\sqrt{3})/Q(q)$

The theorem in this section is from the same fragment [244, pp. 102–104] as the previous theorem, and the proof is very similar.

Entry 11.6.1 (p. 103). *Let*

$$f(q) := \frac{\pi P(q^2)}{2\sqrt{3}Q(q^2)} = \sum_{n=0}^{\infty} \theta_n q^{2n}, \qquad |q| < q_0. \tag{11.6.1}$$

Then, if $n \geq 0$,

$$\theta_n = (-1)^n \frac{3}{G} \sum_{(\lambda)} \frac{h_\lambda(n)}{\lambda^2} e^{n\pi\sqrt{3}/\lambda}, \tag{11.6.2}$$

where λ *runs over the integers of the form* (11.3.4), *and* G *and* $h_\lambda(n)$ *are defined in* (11.3.5) *and* (11.3.8), *respectively. Also, distinct solutions* (c, d) *to* $\lambda = c^2 - cd + d^2$, *which were defined before Entry* 11.3.1, *give rise to distinct terms in the sum in* (11.6.2). *If* $n < 0$, *the sum on the right side of* (11.6.2) *equals* 0.

Proof. By (11.6.1), (11.1.2), (11.1.4), and (11.1.5),

$$\varphi(\tau) = \frac{\pi E_2^*(\tau)}{2\sqrt{3}E_4(\tau)}.$$

By (11.5.10) and (11.3.9) (note that in (11.3.9), $\varphi(\tau) = 1/E_4(\tau)$), we obtain the functional equation

$$\varphi(\tau) = \varphi(V\tau)(c\tau + d)^2 + \frac{\sqrt{3}ci}{E_4(V\tau)}(c\tau + d)^3, \tag{11.6.3}$$

where $V\tau = (a\tau + b)/(c\tau + b)$ is a modular transformation, that is to say, $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. Recall the notation (11.5.14). The analogue of the estimate of φ_1 in (11.5.15) is thus

$$|\varphi(\tau)| \leq |\varphi(V^*\tau)|\,|-k\tau + h|^2 + \frac{\sqrt{3}k}{|E_4(V^*\tau)|}|-k\tau + h|^3. \tag{11.6.4}$$

As with $\varphi_j(\tau)$, the function $\varphi(\tau)$ satisfies the conditions of a modified form of Theorem 11.2.1; here (11.2.1) is replaced by (11.6.3), and $\alpha = \rho = -1/2 + i\sqrt{3}/2$.

In Entry 11.3.1, we applied Theorem 11.2.1 to the modular form $1/E_4(\tau)$. Thus, we can apply (11.2.22) to both φ and $1/E_4(\tau)$, with M replaced by the positive constant $\tilde{M}$ in the latter application, to obtain, by (11.6.3) and (11.2.25),

$$\begin{aligned}
|\varphi(\tau)| &< M|-k\tau + h|^2 + \sqrt{3}k\tilde{M}|-k\tau + h|^3 \\
&= k^2M|\tau - h/k|^2 + \sqrt{3}k^4\tilde{M}|\tau - h/k|^3 \\
&< \frac{4M}{k^2 + kk' + k'^2} + \frac{8\sqrt{3}k\tilde{M}}{(k^2 + kk' + k'^2)^{3/2}}
\end{aligned}$$

$$< \frac{M_0}{k^2 + kk' + k'^2},$$

where $\tilde{M}$ and M_0 are certain positive constants. This implies that the integral in (11.2.5) approaches 0 as m approaches ∞.

We now determine the residues of $\varphi(\tau)$ using the following lemma.

Lemma 11.6.1. *For $\rho = -1/2 + i\sqrt{3}/2$,*

$$\frac{\pi}{2\sqrt{3}} E_2^*(V\rho) = \frac{\pi}{2\sqrt{3}} P(\underline{q}^2) = c^2 - cd + d^2, \tag{11.6.5}$$

where $V\tau = (a\tau + b)/(c\tau + d)$, with a, b, c, and d being integers satisfying $ad - bc = 1$ and

$$\underline{q} = \exp\left(\pi i \left(\frac{a\rho + b}{c\rho + d}\right)\right).$$

Proof. From (11.1.6) and a result of Berndt [51, p. 159],

$$E_2(\rho) = 1 - 24 \sum_{k=1}^{\infty} \frac{(-1)^k k e^{-\pi\sqrt{3}k}}{1 - (-1)^k e^{-\pi\sqrt{3}k}} - \frac{2\sqrt{3}}{\pi} = 0.$$

Thus,

$$0 = E_2(V\rho) = P(\underline{q}^2) - \frac{3}{\pi \text{Im } (V\rho)} = P(\underline{q}^2) - \frac{2\sqrt{3}}{\pi}(c^2 - cd + d^2).$$

This proves (11.6.5). $\square$

We now return to the proof of Entry 11.6.1. By (11.3.12),

$$\text{Res}\left(\frac{1}{E_4(\tau)}, \rho\right) = -\frac{3}{2\pi i G}, \tag{11.6.6}$$

and so, by (11.2.37),

$$\text{Res}\left(\frac{1}{Q(q^2)}, \underline{q}\right) = -\frac{3\underline{q}}{2G(c\rho + d)^6}. \tag{11.6.7}$$

It follows from (11.6.5) and (11.6.7) that

$$\text{Res}\left(f(q), \underline{q}\right) = (c^2 - cd + d^2)\left(-\frac{3\underline{q}}{2G(c\rho + d)^6}\right) = -\frac{3\underline{q}\lambda}{2G(c\rho + d)^6}, \tag{11.6.8}$$

where $\lambda = c^2 - cd + d^2$. By (11.2.38), we then deduce that

$$f(q) = \frac{3}{G} \sum_{(c,d)} \frac{\lambda}{(c\rho + d)^6} \frac{1}{1 - (q/\underline{q})^2}. \tag{11.6.9}$$

Since $\underline{q}$ is the same as in Entry 11.3.1, we have, by (11.3.20) and (11.3.23), for $|q| < e^{-\pi\sqrt{3}/2}$,

$$\begin{aligned} f(q) = \frac{3}{G}\Bigg\{ & \frac{1}{1+e^{\pi\sqrt{3}}q^2} - \frac{3}{3^3}\frac{1}{1+e^{\pi\sqrt{3}/3}q^2} \\ & + \sum_{\substack{(\lambda)\\ \lambda>3}} \left(\frac{\lambda}{(c\rho+d)^6}\frac{1}{1-(q/\underline{q})^2} + \frac{\lambda}{(d\rho+c)^6}\frac{1}{1-(q/\overline{\underline{q}})^2}\right)\Bigg\} \\ = \sum_{n=0}^{\infty} \theta_n q^{2n}, & \end{aligned}$$

where

$$\begin{aligned} \theta_n = \frac{3}{G}\Bigg\{ & (-1)^n e^{n\pi\sqrt{3}} - \frac{(-1)^n}{3^2} e^{n\pi\sqrt{3}/3} \\ & + \sum_{\substack{(\lambda)\\ \lambda>3}} \lambda \left(\frac{1}{(c\rho+d)^6}\underline{q}^{-2n} + \frac{1}{(d\rho+c)^6}\overline{\underline{q}}^{-2n}\right)\Bigg\}. \end{aligned} \tag{11.6.10}$$

In (11.6.10), λ runs over all integers of the form (11.3.4), and, for each fixed λ, the sum is also over all distinct pairs (c, d).

Apart from the factor of λ, the summands in (11.6.10) are the same as those in (11.3.24), and so by (11.3.29), (11.3.31), and (11.3.32), we conclude that

$$\theta_n = (-1)^n \frac{3}{G} \sum_{(\lambda)} \frac{h_\lambda(n)}{\lambda^2} e^{n\pi\sqrt{3}/\lambda},$$

where $h_\lambda(n)$ is defined in (11.3.8). This completes the proof of Entry 11.6.1 for $n \geq 0$.

For $n < 0$, assume $|q| > 1$ and proceed as in the previous proofs. $\square$

11.7 Eight Identities for Eisenstein Series and Theta Functions

Two of the identities considered here involve the classical theta functions (in Ramanujan's notation)

$$\varphi(q) := \sum_{k=-\infty}^{\infty} q^{k^2} \qquad \text{and} \qquad \psi(q) := \sum_{k=0}^{\infty} q^{k(k+1)/2}. \tag{11.7.1}$$

To establish the eight identities, we need to use evaluations of theta functions and Eisenstein series from Chapter 17 of Ramanujan's second notebook [243], [54, pp. 122–138]. If

$$q := \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)}\right), \qquad 0 < x < 1,$$

where ${}_2F_1$ denotes the ordinary hypergeometric function, these evaluations are given in terms of, in Ramanujan's notation,

$$z := {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x), \qquad 0 < x < 1, \tag{11.7.2}$$

and x.

Entry 11.7.1 (pp. 116–117). *Recall that* $Q(q)$ *and* $R(q)$ *are defined by* (11.1.2) *and* (11.1.3)*, respectively, that* $B(q)$ *is defined by* (11.1.10)*, and that* $\varphi(q)$ *and* $\psi(q)$ *are defined in* (11.7.1)*. Then*

(i) $$B(\sqrt{q}) + B(-\sqrt{q}) = 2B(q),$$

(ii) $$B(\sqrt{q})B(-\sqrt{q})B(q) = R(q),$$

(iii) $$Q(q) + 4Q(q^2) = 5B^2(q),$$

(iv) $$-R(q) + 8R(q^2) = 7B^3(q)$$

(v) $$\frac{2}{3}\left(\frac{1}{B(\sqrt{q})} + \frac{1}{B(-\sqrt{q})}\right) - \frac{1}{3B(q)} = \frac{Q(q)}{R(q)},$$

(vi) $$\frac{11}{24}\left(\frac{1}{B(\sqrt{q})} + \frac{1}{B(-\sqrt{q})}\right) + \frac{1}{12B(q)} = \frac{Q(q^2)}{R(q)},$$

(vii) $$-Q(q) + 16Q(q^2) = 15\varphi^8(-q),$$

(viii) $$Q(q) - Q(q^2) = 240q\psi^8(q).$$

Proof of (i). The proof is straightforward, with only the definition of $B(q)$ in (11.1.10) needed in the proof. □

Proof of (ii). By Part (i), with the replacement of $\sqrt{q}$ by q, we can rewrite (ii) in the form

$$B(q)\left\{2B(q^2) - B(q)\right\}B(q^2) = R(q^2). \tag{11.7.3}$$

By (11.1.10), we easily see that

$$2P(q^2) - P(q) = B(q). \tag{11.7.4}$$

From Entries 13(viii), 13(ix), and 13(ii) in Chapter 17 of Ramanujan's second notebook [54, pp. 126–127],

$$B(q) = 2P(q^2) - P(q) = z^2(1 + x), \tag{11.7.5}$$

$$B(q^2) = 2P(q^4) - P(q^2) = z^2(1 - \tfrac{1}{2}x), \tag{11.7.6}$$

$$R(q^2) = z^6(1 + x)(1 - \tfrac{1}{2}x)(1 - 2x), \tag{11.7.7}$$

where z is defined in (11.7.2). Using the evaluations (11.7.5)–(11.7.7) in (11.7.3), we find that each side of (11.7.3) equals

$$z^6(1 + x)(1 - \tfrac{1}{2}x)(1 - 2x),$$

which completes the proof of (ii). □

Proof of (iii). By Entries 13(iii) and 13(i) in Ramanujan's second notebook [54, pp. 126–127],

$$Q(q) = z^4(1+14x+x^2) \quad \text{and} \quad Q(q^2) = z^4(1-x+x^2). \tag{11.7.8}$$

Using (11.7.5) and (11.7.8) in (iii), we find that each side of (iii) reduces to

$$z^4(5+10x+5x^2),$$

thus establishing the truth of (iii). □

Proof of (iv). By Ramanujan's work in Chapter 17 of his second notebook [54, p. 127, Entry 13(iv)],

$$R(q) = z^6(1+x)(1-34x+x^2). \tag{11.7.9}$$

Thus, by (11.7.9), (11.7.5), and (11.7.7), each side of (iv) can be written in the form

$$7z^6(1+x)^3.$$

This completes the proof of (iv). □

Proof of (v). Replacing $\sqrt{q}$ by q and using (i), we can rewrite (v) in the form

$$\frac{2}{3}\left(\frac{1}{B(q)} + \frac{1}{2B(q^2)-B(q)}\right) - \frac{1}{3B(q^2)} = \frac{Q(q^2)}{R(q^2)}. \tag{11.7.10}$$

By (11.7.5) and (11.7.6),

$$\frac{1}{B(q)} + \frac{1}{2B(q^2)-B(q)} = \frac{1}{z^2(1+x)} + \frac{1}{z^2(1-2x)}. \tag{11.7.11}$$

Utilizing (11.7.11) and (11.7.6) and employing a heavy dose of elementary algebra, we find that the left side of (11.7.10) reduces to

$$\frac{1-x+x^2}{z^2(1+x)(1-2x)(1-\frac{1}{2}x)}. \tag{11.7.12}$$

On the other hand, by (11.7.7) and (11.7.8), the right side of (11.7.10) also reduces to (11.7.12). This completes then the proof of (11.7.10), and hence of (v). □

Proof of (vi). Replacing $\sqrt{q}$ by q and using (i), we can rewrite (vi) in the form

$$\frac{11}{24}\left(\frac{1}{B(q)} + \frac{1}{2B(q^2)-B(q)}\right) + \frac{1}{12B(q^2)} = \frac{Q(q^4)}{R(q^2)}. \tag{11.7.13}$$

Using (11.7.11) and (11.7.6), we find that the left side of (11.7.13) takes the shape

$$\frac{1-x+\frac{1}{16}x^2}{z^2(1+x)(1-2x)(1-\frac{1}{2}x)}. \tag{11.7.14}$$

Again, from Ramanujan's work [54, p. 127, Entry 13(v)],

$$Q(q^4)=z^4(1-x+\tfrac{1}{16}x^2), \tag{11.7.15}$$

so that, by (11.7.15) and (11.7.7), the right side of (11.7.13) also reduces to (11.7.14). This completes the proof of (vi). □

Proof of (vii). By Entry 10(ii) in Chapter 17 of Ramanujan's second notebook [54, p. 122],

$$\varphi(-q)=\sqrt{z}(1-x)^{1/4}. \tag{11.7.16}$$

Thus the left side of (vii) takes the shape

$$15z^4(1-x)^2, \tag{11.7.17}$$

while by (11.7.8) the right side of (vii) also equals (11.7.17). □

Proof of (viii). Appealing again to Chapter 17 of Ramanujan's second notebook [54, p. 123, Entry 11(i)], we have

$$\psi(q)=\sqrt{\frac{1}{2}z}(x/q)^{1/8}. \tag{11.7.18}$$

Thus, by (11.7.18), the left side of (viii) equals

$$240(\tfrac{1}{16}z^4x)=15z^4x, \tag{11.7.19}$$

while, by (11.7.8), the right side of (viii) equals (11.7.19) as well. □

11.8 The Coefficients of $1/B(q)$

Define the coefficients b_n by

$$\frac{1}{B(q)}=\sum_{n=0}^{\infty}b_nq^n, \tag{11.8.1}$$

where $|q|<q_0<1$, for q_0 sufficiently small.

We are now ready to state the first main theorem about $B(q)$, which establishes an assertion of Ramanujan from his letter to Hardy containing Entry 11.4.1 [244, p. 117], [74, p. 190]. For the sake of brevity, we write

$$\delta_n=\sum_{(\mu)}V_\mu(n), \tag{11.8.2}$$

where

$$V_\mu(n)=\frac{2}{J}\frac{v_\mu(n)}{\mu^2}e^{2n\pi/\mu}, \tag{11.8.3}$$

where $v_\mu(n)$ is defined by (11.4.8) and (11.4.9).

Entry 11.8.1 (p. 117). *Recall that the coefficients* b_n *are defined by* (11.8.1). *Then, with* $V_\mu(n)$ *defined by* (11.8.3),

$$b_n = -3 \sum_{(\mu_e)} V_{\mu_e}(n), \tag{11.8.4}$$

where μ_e *runs over the even values of* μ. *In other words,* μ_e *runs over the even integers of the form* (11.4.4).

Set $\beta(\tau) = B(q)$, where $q = \exp(2\pi i\tau)$. Then $\beta(\tau)$ is a modular form on $\Gamma_0(2)$, as we show in the next lemma. We remark that $\beta(\tau)$ is not a modular form on $\Gamma(1)$, because the dimension of the space of modular forms of weight 2 with multiplier system identically equal to 1 on $\Gamma(1)$ is zero [246, p. 103].

Lemma 11.8.1. *The function* $\beta(\tau)$ *is a modular form of weight* 2 *and multiplier system identically equal to* 1 *on the group* $\Gamma_0(2)$. *That is,*

$$\beta\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\beta(\tau),$$

where a, b, c, $d \in \mathbb{Z}$; $ad-bc=1$; *and* c *is even.*

Proof. Recall that $E_2^*(\tau)$ is defined by (11.1.5). Thus, by (11.7.4),

$$2E_2^*(2\tau) - E_2^*(\tau) = \beta(\tau). \tag{11.8.5}$$

Recall [255, pp. 50, 68] that for any modular transformation $(a\tau+b)/(c\tau+d)$,

$$E_2^*\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2 E_2^*(\tau) - \frac{6ci}{\pi}(c\tau+d), \tag{11.8.6}$$

and so

$$E_2^*\left(2\frac{a\tau+b}{c\tau+d}\right) = E_2^*\left(\frac{a(2\tau)+2b}{\dfrac{c}{2}(2\tau)+d}\right) = (c\tau+d)^2 E_2^*(2\tau) - \frac{3ci}{\pi}(c\tau+d), \tag{11.8.7}$$

for c even. Thus, by (11.8.6) and (11.8.7),

$$2E_2^*\left(2\frac{a\tau+b}{c\tau+d}\right) - E_2^*\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\left(2E_2^*(2\tau) - E_2^*(\tau)\right). \tag{11.8.8}$$

By (11.8.8) and (11.8.5), we complete the proof. □

Lemma 11.8.2. *The function* $1/\beta(\tau)$ *has a simple pole at* $\tau = \frac{1}{2}(1+i)$, *and this is the only pole in a fundamental region of* $\Gamma_0(2)$.

By Lemmas 11.8.1 and 11.8.2, it follows that $1/\beta(\tau)$ has poles at the points $\tau = (a\frac{1}{2}(1+i)+b)/(c\frac{1}{2}(1+i)+d)$, where a, b, and d are integers, c is an even integer, and $ad-bc=1$. By the valence formula [246, p. 98], there are no further poles of $1/\beta(\tau)$ in a fundamental region of $\Gamma_0(2)$.

Proof of Lemma 11.8.2. By (11.8.5) and (11.1.6), we easily see that

$$\frac{1}{\beta(\tau)} = \frac{1}{2E_2(2\tau) - E_2(\tau)}. \tag{11.8.9}$$

We show that both functions in the denominator of (11.8.9) vanish at $\tau = \frac{1}{2}(1+i)$. First, by (11.5.21),

$$E_2(i) = 0. \tag{11.8.10}$$

Thus, by periodicity,

$$E_2\left(2\frac{1+i}{2}\right) = E_2(1+i) = E_2(i) = 0,$$

and, since $E_2(\tau)$ satisfies the functional equation of a modular form of weight 2,

$$E_2\left(\frac{1+i}{2}\right) = E_2\left(\frac{i}{1+i}\right) = E_2\left(\frac{0+1i}{1+1i}\right) = (1+i)^2 E_2(i) = 0.$$

Thus, both terms in the denominator of (11.8.9) vanish, and the proof of the lemma is therefore complete. □

The main theorem of Hardy and Ramanujan, Theorem 11.2.1, is therefore inapplicable. However, Poincaré [176, pp. 210–215, 432–462, 606–614, 618 (paragraph 2)], Lehner [201], and particularly Petersson [228, pp. 460–461, Satz 3], [229], [230] have extensively generalized Hardy and Ramanujan's theorem. We only need the special case for the subgroup $\Gamma_0(2)$, which we state below.

Theorem 11.8.1. *Suppose that $f(q) = f(e^{\pi i\tau}) = \phi(\tau)$ is analytic for $q = 0$, is meromorphic in the unit circle, and satisfies the functional equation*

$$\phi(\tau) = \phi\left(\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^n, \tag{11.8.11}$$

where $a, b, c, d \in \mathbb{Z}$; $ad - bc = 1$; c is even; and $n \in \mathbb{Z}^+$. If $\phi(\tau)$ has only one pole in a fundamental region for $\Gamma_0(2)$, a simple pole at $\tau = \alpha$ with residue A, then

$$f(q) = -2\pi i A \sum \frac{1}{(c\alpha+d)^{n+2}} \frac{1}{1-(q/\underline{q})^2}, \qquad |q| < q_0, \tag{11.8.12}$$

where

$$\underline{q} = \exp\left(\left(\frac{a\alpha+b}{c\alpha+d}\right)\pi i\right), \tag{11.8.13}$$

and the summation runs over all pairs of coprime integers (c, d) (with c even) that yield distinct values for the set $\{\underline{q}, -\underline{q}\}$, and a and b are any integral solutions of

$$ad - bc = 1. \tag{11.8.14}$$

We are now prepared to prove Entry 11.8.1.

Proof of Entry 11.8.1. Using (11.7.4) and (11.1.7), we find that

$$\begin{aligned}
B'(q) &= 4qP'(q^2) - P'(q) \\
&= \frac{P^2(q^2) - Q(q^2)}{3q} - \frac{P^2(q) - Q(q)}{12q} \\
&= \frac{1}{12q}\left\{\left(2P(q^2) - P(q)\right)\left(2P(q^2) + P(q)\right) - 4Q(q^2) + Q(q)\right\} \\
&= \frac{1}{12q}\left\{B(q)\left(2P(q^2) + P(q)\right) - 4Q(q^2) + Q(q)\right\}. \qquad (11.8.15)
\end{aligned}$$

By Lemma 11.8.2, $B(-e^{-\pi}) = 0$, and therefore from Entry 11.7.1 (iii), we deduce that

$$Q(-e^{-\pi}) = -4Q(e^{-2\pi}). \qquad (11.8.16)$$

Hence, setting $q = -e^{-\pi}$ in (11.8.15), we find that

$$B'(-e^{-\pi}) = \frac{1}{12(-e^{-\pi})}\left\{-8Q(e^{-2\pi})\right\} = \frac{2}{3}e^{\pi}Q(e^{-2\pi}), \qquad (11.8.17)$$

which is explicitly calculated in Proposition 11.8.2 below.

We now apply Theorem 11.8.1 to $\phi(\tau) = f(q) := 1/B(q^2)$, where $q = e^{\pi i\tau}$ and $\alpha := (1+i)/2$. By the chain rule,

$$A = \operatorname{Res}(\phi, \alpha) = \frac{\operatorname{Res}(f, e^{\pi i\alpha})}{\pi i e^{\pi i\alpha}}, \qquad (11.8.18)$$

and, by (11.8.17),

$$\begin{aligned}
\operatorname{Res}(f, e^{\pi i\alpha}) &= \operatorname{Res}\left(\frac{1}{B(q^2)}, e^{\pi i\alpha}\right) = \left.\frac{1}{dB(q^2)/dq}\right|_{q=e^{\pi i\alpha}} \\
&= \left.\frac{1}{2qB'(q^2)}\right|_{q=e^{\pi i\alpha}} = \frac{1}{2e^{\pi i\alpha}}\frac{3e^{-\pi}}{2Q(e^{-2\pi})}. \qquad (11.8.19)
\end{aligned}$$

Hence, combining (11.8.18) and (11.8.19), we deduce that

$$-2\pi i A = -\frac{2\pi i}{\pi i e^{\pi i\alpha}}\frac{3e^{-\pi}}{4e^{\pi i\alpha}Q(e^{-2\pi})} = \frac{3}{2Q(e^{-2\pi})}. \qquad (11.8.20)$$

We next calculate $\underline{q}$. Recall that $ad - bc = 1$ with c even. Thus, d is odd. Hence,

$$\begin{aligned}
\underline{q} &= \exp\left(\pi i\frac{a\alpha + b}{c\alpha + d}\right) = \exp\left(\pi i\frac{a + 2b + ai}{c + 2d + ci}\right) \\
&= \exp\left(\pi i\frac{(a + 2b + ai)(c + 2d - ci)}{c^2 + 4cd + 4d^2 + c^2}\right)
\end{aligned}$$

$$= \exp\left(\pi i \frac{(a+b)(c+d)+bd+i}{(c+d)^2+d^2}\right).$$

Let $\mu = c'^2 + d^2$, where $c' = c+d$, and let $a' = a+b$. Note that $a'd - bc' = 1$ and that μ is even. Thus,

$$\underline{q} = \exp\left(\pi i \frac{a'c'+bd+i}{\mu}\right) = \exp\left(-\frac{\pi}{\mu}\right)\exp\left(\pi i \frac{a'c'+bd}{\mu}\right). \tag{11.8.21}$$

Next,

$$c\alpha + d = c\frac{1}{2}(1+i) + d = \frac{1}{1+i}(ci+d+di) = \frac{1}{1+i}(c'i+d), \tag{11.8.22}$$

where $c' = c+d$.

The requisite calculations have now been made in order to apply Theorem 11.8.1. By (11.8.1), (11.8.20), and (11.8.22), we deduce that

$$\begin{aligned}\sum_{n=0}^{\infty} b_n q^{2n} &= \frac{1}{B(q^2)} = \frac{3}{2Q(e^{-2\pi})}\sum_{(c',d)} \frac{1}{(1+i)^{-4}(c'i+d)^4}\frac{1}{1-(q/\underline{q})^2} \\ &= -\frac{6}{Q(e^{-2\pi})}\sum_{(c',d)} \frac{1}{(c'i+d)^4}\frac{1}{1-(q/\underline{q})^2},\end{aligned} \tag{11.8.23}$$

where the sum is over all pairs c', d with $a'd - bc' = 1$ and c' odd (since c is even and d is odd), and where $\underline{q}$ is given by (11.8.21). Now from (11.4.14),

$$\sum_{n=0}^{\infty} \delta_n q^{2n} = \frac{Q(q^2)}{R(q^2)} = \frac{2}{Q(e^{-2\pi})}\sum_{(c,d)} \frac{1}{(ci+d)^4}\frac{1}{1-(q/\underline{q})^2}, \tag{11.8.24}$$

where the sum is over *all* pairs c, d with $ad - bc = 1$, and where

$$\underline{q} = \exp\left(\pi i \frac{ai+b}{ci+d}\right) = \exp\left(-\frac{\pi}{\mu}\right)\exp\left(\pi i \frac{ac+bd}{\mu}\right).$$

A comparison of (11.8.23) and (11.8.24) shows that the right sides of (11.8.23) and (11.8.24) are identical except in two respects. First, in (11.8.23), there is an extra factor of -3 on the right side. Second, upon expanding the summands in geometric series on the right sides of (11.8.23) and (11.8.24), we see that the sum in (11.8.23) is over only even μ. In other words,

$$b_n = -3\sum_{(\mu_e)} V_{\mu_e}(n),$$

where $V_{\mu_e}(n)$ is defined by (11.8.3). This completes the proof. □

The series in (11.8.4) converges very rapidly to b_n. Using *Mathematica*, we calculated $b_n, 1 \leq n \leq 10$, and the first two terms of (11.8.4). As the following table shows, only two terms of the series give extraordinary approximations.

n	b_n	$-3(V_2(n) + V_{10}(n))$
1	-24	-23.971586
2	552	551.992987
3	$-12{,}768$	$-12{,}768.016604$
4	295,464	295,463.973727
5	$-6{,}837{,}264$	$-6{,}837{,}264.003874$
6	158,219,040	158,219,040.077478
7	$-3{,}661{,}298{,}112$	$-3{,}661{,}298{,}112.002170$
8	84,724,974,120	84,724,974,119.926326
9	$-1{,}960{,}594{,}584{,}504$	$-1{,}960{,}594{,}584{,}504.044556$
10	45,369,516,658,032	45,369,516,658,031.999703

Using Entry 11.7.1 (v), we can easily establish a formula for δ_n in terms of b_n, but we were unable to use this relation to prove Entry 11.8.1.

Proposition 11.8.1. *For each positive integer n,*

$$\delta_n = \frac{4}{3}b_{2n} - \frac{1}{3}b_n.$$

Proof. By Entry 11.7.1(v), (11.4.5), and (11.8.1),

$$\begin{aligned}
\sum_{n=0}^{\infty} \delta_n q^n = \frac{Q(q)}{R(q)} &= \frac{2}{3}\left(\frac{1}{B(\sqrt{q})} + \frac{1}{B(-\sqrt{q})}\right) - \frac{1}{3B(q)} \\
&= \frac{2}{3}\left(\sum_{n=0}^{\infty} b_n(\sqrt{q})^n + \sum_{n=0}^{\infty} b_n(-\sqrt{q})^n\right) - \frac{1}{3}\sum_{n=0}^{\infty} b_n q^n \\
&= \frac{4}{3}\sum_{n=0}^{\infty} b_{2n} q^n - \frac{1}{3}\sum_{n=0}^{\infty} b_n q^n.
\end{aligned}$$

□

In our previous attempts to prove Entry 11.8.1, we showed that $V_\mu(n)$, for μ odd, is a multiple of $V_\mu(2n)$ when μ is even. Although we were not able to use this result in our goal, we think the formula is very interesting by itself and so prove it now.

Theorem 11.8.2. *For each positive integer n,*

$$V_{\mu_o}(n) = -4V_{2\mu_o}(2n), \tag{11.8.25}$$

where μ_o is an odd integer of the form (11.4.4).

Proof. We first easily establish the case $\mu_o = 1$. From (11.8.3) and (11.4.8), we see that

$$V_1(n) = \frac{2}{J} e^{2n\pi},$$

while

$$-4V_2(2n) = \frac{-8}{J} \frac{(-1)^{2n+1}}{2^2} e^{4n\pi/2} = \frac{2}{J} e^{2n\pi}.$$

Now we assume that $\mu_o > 1$. Suppose that (a_1, b_1, c_1, d_1) satisfies

$$ad - bc = 1, \tag{11.8.26}$$

with $c^2 + d^2 = \mu_o$. We can assume, without loss of generality, that $a_1 > b_1 > 0$ and that a_1 and b_1 are odd. Since

$$(c_1 + d_1)^2 + (c_1 - d_1)^2 = 2\mu_o,$$

we find that if we let

$$c_2 = c_1 + d_1 \qquad \text{and} \qquad d_2 = c_1 - d_1, \tag{11.8.27}$$

then (c_2, d_2) satisfies

$$c^2 + d^2 = 2\mu_o.$$

Similarly, if we let

$$a_2 = -\frac{1}{2}(a_1 + b_1) \qquad \text{and} \qquad b_2 = -\frac{1}{2}(a_1 - b_1), \tag{11.8.28}$$

then (a_2, b_2, c_2, d_2) satisfies (11.8.26). By (11.8.3) and (11.4.9), in order to prove (11.8.25), it suffices to prove that

$$\begin{aligned}
&\frac{2}{J} \frac{2\cos\left\{(a_1c_1 + b_1d_1)\dfrac{2n\pi}{\mu_o} + 4\tan^{-1}\dfrac{c_1}{d_1}\right\}}{\mu_0^2} e^{2n\pi/\mu_o} \\
&\qquad = -\frac{8}{J} \frac{2\cos\left\{(a_2c_2 + b_2d_2)\dfrac{4n\pi}{2\mu_o} + 4\tan^{-1}\dfrac{c_2}{d_2}\right\}}{(2\mu_o)^2} e^{4n\pi/(2\mu_o)},
\end{aligned}$$

or equivalently that

$$\begin{aligned}
&\cos\left\{(a_1c_1 + b_1d_1)\frac{2n\pi}{\mu_o} + 4\tan^{-1}\frac{c_1}{d_1}\right\} \\
&\qquad = -\cos\left\{(a_2c_2 + b_2d_2)\frac{2n\pi}{\mu_o} + 4\tan^{-1}\frac{c_2}{d_2}\right\}.
\end{aligned} \tag{11.8.29}$$

By the identities (11.8.27) and (11.8.28), we can rewrite the right-hand side as

$$-\cos\left\{\left(-\frac{1}{2}(a_1+b_1)(c_1+d_1)-\frac{1}{2}(a_1-b_1)(c_1-d_1)\right)\frac{2n\pi}{\mu_o}\right.$$
$$\left.+4\tan^{-1}\frac{c_1+d_1}{c_1-d_1}\right\}$$
$$=-\cos\left\{(-a_1c_1-b_1d_1)\frac{2n\pi}{\mu_o}+4\tan^{-1}\frac{c_1+d_1}{c_1-d_1}\right\}. \qquad (11.8.30)$$

Note that

$$\operatorname{Arctan}\frac{c_1+d_1}{c_1-d_1}=\frac{k\pi}{4}-\operatorname{Arctan}\frac{c_1}{d_1}, \qquad (11.8.31)$$

where $k\equiv 3 \pmod 4$. Using (11.8.31), we can rewrite the latter expression in (11.8.30) as

$$-\cos\left\{-(a_1c_1+b_1d_1)\frac{2n\pi}{\mu_o}-4\tan^{-1}\frac{c_1}{d_1}+k\pi\right\}$$
$$=\cos\left\{(a_1c_1+b_1d_1)\frac{2n\pi}{\mu_o}+4\tan^{-1}\frac{c_1}{d_1}\right\}.$$

Thus, (11.8.29) has been proved, and hence (11.8.25) as well. □

We close this section by showing that $Q(e^{-2\pi})$ in Entry 11.4.1 can be evaluated in closed form. Then in a corollary, we evaluate another interesting series.

Proposition 11.8.2. *We have*

$$Q(e^{-2\pi})=\frac{3\pi^2}{4\Gamma^8(\frac{3}{4})}. \qquad (11.8.32)$$

Proof. We apply Entry 13(i) in Chapter 17 of Ramanujan's second notebook [54, p. 126]. In [54], $M(q)=Q(q)$, in the present notation. Set $y=\pi$ there, and note that $x=\frac{1}{2}$. We find immediately that

$$Q(e^{-2\pi})=\tfrac{3}{4}\,{}_2F_1^4(\tfrac{1}{2},\tfrac{1}{2};1;\tfrac{1}{2})=\frac{3\pi^2}{4\Gamma^8(\frac{3}{4})},$$

where we have employed a special case of a well-known theorem of Gauss, rediscovered by Ramanujan [53, p. 42, Entry 34]. (See also [54, p. 103, equation (6.15)].) Hence, (11.8.32) has been shown. □

Corollary 11.8.1. *We have*

$$\sum_{n=0}^{\infty}\frac{(2n+1)^2}{\cosh^2\{(2n+1)\pi/2\}}=\frac{\pi^2}{12\Gamma^8(\frac{3}{4})}.$$

Proof. From the definition of $B(q)$ in (11.1.10), we easily find that

$$B'(q) = 24 \sum_{n=0}^{\infty} \frac{(2n+1)^2 q^{2n}}{(1-q^{2n+1})^2}.$$

Setting $q = -e^{-\pi}$ and using (11.8.17), we find that

$$\begin{aligned}\frac{2}{3} e^{\pi} Q(e^{-2\pi}) = B'(-e^{-\pi}) &= 24 e^{\pi} \sum_{n=0}^{\infty} \frac{(2n+1)^2 e^{-(2n+1)\pi}}{(1+e^{-(2n+1)\pi})^2} \\ &= 6 e^{\pi} \sum_{n=0}^{\infty} \frac{(2n+1)^2}{\cosh^2\{(2n+1)\pi/2\}}.\end{aligned}$$

If we now use (11.8.32) on the left side above and simplify, we complete the proof. □

Corollary 11.8.1 might be compared with further explicit evaluations of series containing the hyperbolic function cosh given by Ramanujan in Entry 16 of Chapter 17 in his second notebook [54, p. 134], in Entry 6 of Chapter 18 in his second notebook [54, p. 153], and in two particular results in his first notebook [57, pp. 398, 402, Entries 76, 78].

11.9 Formulas for the Coefficients of Further Eisenstein Series

Ramanujan [244, pp. 117–118], [74, pp. 190–191] concludes his letter to Hardy with three identities that are similar to (11.8.4). We show how each of the identities follows from (11.8.4), but first we need to make several definitions.

Let, for $|q| < q_0$,

$$\frac{Q(q^2)}{R(q)} =: \sum_{n=0}^{\infty} \sigma_n q^n, \quad \frac{\varphi^8(-q)}{R(q)} =: \sum_{n=0}^{\infty} v_n q^n, \quad \text{and} \quad \frac{q\psi^8(q)}{R(q)} =: \sum_{n=0}^{\infty} \chi_n q^n. \tag{11.9.1}$$

Entry 11.9.1 (pp. 117–118). *Suppose that* (11.8.4) *holds. Then*

(i) $$\sigma_n = \frac{11}{16} \sum_{(\mu_o)} V_{\mu_o}(n) - \frac{1}{4} \sum_{(\mu_e)} V_{\mu_e}(n),$$

(ii) $$v_n = \frac{2}{3} \sum_{(\mu_o)} V_{\mu_o}(n) - \frac{1}{3} \sum_{(\mu_e)} V_{\mu_e}(n),$$

(iii) $$\chi_n = \frac{1}{768} \sum_{(\mu_o)} V_{\mu_o}(n) + \frac{1}{192} \sum_{(\mu_e)} V_{\mu_e}(n),$$

where μ_o and μ_e run over the odd and even integers of the form (11.4.4), *respectively.*

Proof of (i). By (11.9.1), Entry 11.7.1(vi), (v), (11.4.5), and (11.8.1),

$$\begin{aligned}\sum_{n=0}^{\infty}\sigma_n q^n &= \frac{11}{24}\left(\frac{1}{B(\sqrt{q})}+\frac{1}{B(-\sqrt{q})}\right)+\frac{1}{12B(q)}\\ &= \frac{11}{16}\left(\frac{2}{3}\left(\frac{1}{B(\sqrt{q})}+\frac{1}{B(-\sqrt{q})}\right)-\frac{1}{3B(q)}\right)+\frac{5}{16}\frac{1}{B(q)}\\ &= \frac{11}{16}\frac{Q(q)}{R(q)}+\frac{5}{16}\frac{1}{B(q)}\\ &= \frac{11}{16}\sum_{n=0}^{\infty}\delta_n q^n+\frac{5}{16}\sum_{n=0}^{\infty}b_n q^n.\end{aligned}$$

Thus, by (11.8.2) and (11.8.4),

$$\begin{aligned}\sigma_n &= \frac{11}{16}\delta_n+\frac{5}{16}b_n\\ &= \frac{11}{16}\sum_{(\mu)}V_{\mu}(n)+\frac{5}{16}\left(-3\sum_{(\mu_e)}V_{\mu_e}(n)\right)\\ &= \frac{11}{16}\sum_{(\mu_o)}V_{\mu_o}(n)-\frac{1}{4}\sum_{(\mu_e)}V_{\mu_e}(n).\end{aligned}$$

□

Proof of (ii). By (11.9.1), Entry 11.7.1(vii), and (11.4.5),

$$\sum_{n=0}^{\infty}\upsilon_n q^n = -\frac{1}{15}\frac{Q(q)}{R(q)}+\frac{16}{15}\frac{Q(q^2)}{R(q)} = -\frac{1}{15}\sum_{n=0}^{\infty}\delta_n q^n+\frac{16}{15}\sum_{n=0}^{\infty}\sigma_n q^n.$$

Thus, by (11.8.2) and part (i),

$$\begin{aligned}\upsilon_n &= -\frac{1}{15}\delta_n+\frac{16}{15}\sigma_n\\ &= -\frac{1}{15}\sum_{(\mu)}V_{\mu}(n)+\frac{16}{15}\left(\frac{11}{16}\sum_{(\mu_o)}V_{\mu_o}(n)-\frac{1}{4}\sum_{(\mu_e)}V_{\mu_e}(n)\right)\\ &= \frac{2}{3}\sum_{(\mu_o)}V_{\mu_o}(n)-\frac{1}{3}\sum_{(\mu_e)}V_{\mu_e}(n).\end{aligned}$$

□

Proof of (iii). By (11.9.1), Entry 11.7.1(viii), and (11.4.5),

$$\sum_{n=0}^{\infty}\chi_n q^n = \frac{1}{240}\frac{Q(q)}{R(q)}-\frac{1}{240}\frac{Q(q^2)}{R(q)} = \frac{1}{240}\sum_{n=0}^{\infty}\delta_n q^n-\frac{1}{240}\sum_{n=0}^{\infty}\sigma_n q^n.$$

Thus, by (11.8.2) and part (i),

$$
\begin{aligned}
\chi_n &= \frac{1}{240}\delta_n - \frac{1}{240}\sigma_n \\
&= \frac{1}{240}\sum_{(\mu)} V_\mu(n) - \frac{1}{240}\left(\frac{11}{16}\sum_{(\mu_o)} V_{\mu_o}(n) - \frac{1}{4}\sum_{(\mu_e)} V_{\mu_e}(n)\right) \\
&= \frac{1}{768}\sum_{(\mu_o)} V_{\mu_o}(n) + \frac{1}{192}\sum_{(\mu_e)} V_{\mu_e}(n)\,.
\end{aligned}
$$

□

11.10 The Coefficients of $1/B^2(q)$

In another letter to Hardy [244, pp. 105–109], [74, pp. 185–188], Ramanujan offers a formula for the coefficients of $1/B^2(q)$. By Lemma 11.8.2, $1/\beta^2(\tau)$ has a double pole at $(1+i)/2$. To the best of our knowledge, the generalizations of the principal theorem of Hardy and Ramanujan [177, Theorem 1], [242, p. 312] that we cited earlier do not consider double poles, mainly because of calculational difficulties. In fact, after stating his main theorem, Lehner [201, p. 65, Theorem 1] writes, "Poles of higher order can be treated in an analogous manner, but the algebraic details, into which we do not enter here, become rather complicated." Since $1/\beta^2(\tau)$ has only one double pole on a fundamental region for $\Gamma_0(2)$, we confine ourselves to stating our theorem for $\Gamma_0(2)$ only and proving it for modular forms with only one double pole on a fundamental region for $\Gamma_0(2)$.

Theorem 11.10.1. *Suppose that $f(q) = f(e^{\pi i\tau}) = \phi(\tau)$ is analytic for $q = 0$, is meromorphic in the unit circle, and satisfies the functional equation*

$$\phi(\tau) = \phi\left(\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^n, \tag{11.10.1}$$

where $a, b, c, d \in \mathbb{Z}$; $ad - bc = 1$; c is even; and $n \in \mathbb{Z}^+$. Assume that $\phi(\tau)$ has only one pole in a fundamental region for $\Gamma_0(2)$, a double pole at $\tau = \alpha$. Suppose that $f(q)$ and $\phi(\tau)$ have the Laurent expansions,

$$\phi(\tau) = \frac{r_2}{(\tau-\alpha)^2} + \frac{r_1}{\tau-\alpha} + \cdots = \frac{\ell_2}{(q-e^{\pi i\alpha})^2} + \frac{\ell_1}{q-e^{\pi i\alpha}} + \cdots = f(q). \tag{11.10.2}$$

Then

$$f(q) = 2\pi i\sum_{c,d}\left\{\frac{cr_2(n+2)}{(c\alpha+d)^{n+3}} - \frac{r_1}{(c\alpha+d)^{n+2}}\right\}\frac{1}{1-(q/\underline{q})^2}$$

$$- 4\pi^2 r_2 \sum_{c,d} \frac{1}{(c\alpha+d)^{n+4}} \frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2}, \qquad |q| < q_0, \tag{11.10.3}$$

where

$$\underline{q} = \exp\left(\left(\frac{a\alpha+b}{c\alpha+d}\right)\pi i\right), \tag{11.10.4}$$

and the summation runs over all pairs of coprime integers (c,d) (*with* c *even*) *that yield distinct values for the set* $\{\underline{q}, -\underline{q}\}$, *and* a *and* b *are any integral solutions of*

$$ad - bc = 1. \tag{11.10.5}$$

Furthermore,

$$r_1 = -\frac{i\ell_1}{\pi e^{\pi i\alpha}} + \frac{i\ell_2}{\pi e^{2\pi i\alpha}} \qquad \textit{and} \qquad r_2 = -\frac{\ell_2}{\pi^2 e^{2\pi i\alpha}}. \tag{11.10.6}$$

Proof. For brevity, set

$$T := T(\tau) := \frac{a\tau+b}{c\tau+d} \qquad \text{and} \qquad Z := \frac{a\alpha+b}{c\alpha+d}. \tag{11.10.7}$$

We want to calculate the Laurent expansion of ϕ as a function of T in a neighborhood of Z. Since

$$\tau = \frac{dT-b}{-cT+a}, \tag{11.10.8}$$

we easily find that

$$\begin{aligned}\tau - \alpha &= \frac{T(d+c\alpha)-(b+a\alpha)}{-cT+a} = \frac{T - \dfrac{a\alpha+b}{c\alpha+d}}{-cT+a}(c\alpha+d)\\ &= \frac{T-Z}{-cT+a}(c\alpha+d) = \frac{T-Z}{-cZ+a-c(T-Z)}(c\alpha+d).\end{aligned} \tag{11.10.9}$$

However, by (11.10.7) and (11.10.5), we easily find that

$$- cZ + a = \frac{1}{c\alpha+d}. \tag{11.10.10}$$

Employing (11.10.10) in (11.10.9), we find that

$$\tau - \alpha = \frac{T-Z}{1-c(c\alpha+d)(T-Z)}(c\alpha+d)^2,$$

or

$$\frac{1}{\tau-\alpha} = \frac{1-c(c\alpha+d)(T-Z)}{(T-Z)(c\alpha+d)^2} \tag{11.10.11}$$

and

$$\frac{1}{(\tau-\alpha)^2}=\frac{1-2c(c\alpha+d)(T-Z)+c^2(c\alpha+d)^2(T-Z)^2}{(T-Z)^2(c\alpha+d)^4}. \tag{11.10.12}$$

We next seek the expansion of $(c\tau+d)^n$ in powers of $(T-Z)$. By (11.10.8), (11.10.5), and (11.10.10),

$$\begin{aligned}(c\tau+d)^n &= (-cT+a)^{-n}\\ &= ((-cZ+a)-c(T-Z))^{-n}\\ &= \left(\frac{1}{c\alpha+d}-c(T-Z)\right)^{-n}\\ &= (c\alpha+d)^n\left(1-c(c\alpha+d)(T-Z)\right)^{-n}. \end{aligned} \tag{11.10.13}$$

Thus, from (11.10.2), (11.10.11)–(11.10.13), and (11.10.1),

$$\begin{aligned}\phi(\tau) &= \left(\frac{1}{(c\alpha+d)^4(T-Z)^2}-\frac{2c}{(c\alpha+d)^3(T-Z)}+\frac{c^2}{(c\alpha+d)^2}\right)r_2\\ &\quad+\left(\frac{1}{(c\alpha+d)^2(T-Z)}-\frac{c}{c\alpha+d}\right)r_1+\cdots\\ &= (c\tau+d)^n\phi(T)\\ &= (c\alpha+d)^n\left(1-c(c\alpha+d)(T-Z)\right)^{-n}\left(\frac{R_2}{(T-Z)^2}+\frac{R_1}{T-Z}+\cdots\right), \end{aligned} \tag{11.10.14}$$

where R_1 and R_2 are the coefficients in the principal part of $\phi(T)$ about Z. Thus, rearranging (11.10.14), we easily find that

$$\begin{aligned}\frac{R_2}{(T-Z)^2}+\frac{R_1}{T-Z}+\cdots &= (c\alpha+d)^{-n}\left(1-cn(c\alpha+d)(T-Z)+\cdots\right)\\ &\quad\times\left(\left(\frac{1}{(c\alpha+d)^4(T-Z)^2}-\frac{2c}{(c\alpha+d)^3(T-Z)}+\frac{c^2}{(c\alpha+d)^2}\right)r_2\right.\\ &\quad\left.+\left(\frac{1}{(c\alpha+d)^2(T-Z)}-\frac{c}{c\alpha+d}\right)r_1+\cdots\right). \end{aligned} \tag{11.10.15}$$

It follows from above that

$$R_1=(c\alpha+d)^{-n}\left(-\frac{2cr_2}{(c\alpha+d)^3}-\frac{cnr_2}{(c\alpha+d)^3}+\frac{r_1}{(c\alpha+d)^2}\right) \tag{11.10.16}$$

and

$$R_2=\frac{r_2}{(c\alpha+d)^{n+4}}. \tag{11.10.17}$$

We now proceed as in [177, Theorem 1], [242, p. 312], or as in the proof of Theorem 11.2.1. Recall that the definition of $\underline{q}$ is given in (11.8.13). Accordingly,

$$f(q) = -\sum \operatorname{Res}\left(\frac{f(z)}{z-q}, \pm\underline{q}\right), \tag{11.10.18}$$

where the sum is over all poles $\pm\underline{q}$. If

$$g(z) := \frac{1}{z-q},$$

then, by Taylor's theorem,

$$g(z) = \frac{1}{\pm\underline{q} - q} - \frac{1}{(\pm\underline{q} - q)^2}(z \mp \underline{q}) + \cdots . \tag{11.10.19}$$

Let us write $f(z)$ as

$$f(z) = \frac{P_2}{(z-z_0)^2} + \frac{P_1}{z-z_0} + \cdots .$$

Then

$$\operatorname{Res}\left(\frac{f(z)}{z-q}, z_0\right) = P_2\left(\frac{1}{z-q}\right)'\bigg|_{z=z_0} + P_1 \frac{1}{z-q}\bigg|_{z=z_0}. \tag{11.10.20}$$

We need to find P_1 and P_2 for $z_0 = \pm\underline{q}$.

Next, we take the Laurent expansion (11.10.15) and convert it into a Laurent expansion in powers of $(z - \underline{q})$. Observe that $e^{\pi i(Z+1)} = -\underline{q}$. Thus, the Laurent expansion in powers of $(z+\underline{q})$ arises from (11.10.15) with Z replaced by $Z+1$. Since the arguments in the two cases $+\underline{q}$ and $-\underline{q}$ are identical, we consider only the poles $+\underline{q}$. Set $z = e^{\pi i \tau}$ and recall that $\underline{q} = e^{\pi i Z}$. Also, put

$$h(z) = \log z - \log \underline{q},$$

where the principal branch of log is chosen. Then, by Taylor's theorem,

$$h(z) = \frac{1}{\underline{q}}(z-\underline{q}) - \frac{1}{2\underline{q}^2}(z-\underline{q})^2 + \cdots ,$$

and so

$$\frac{1}{h(z)} = \frac{\underline{q}}{z-\underline{q}}\left(1 + \frac{1}{2\underline{q}}(z-\underline{q}) + \cdots\right)$$

and

$$\frac{1}{h^2(z)} = \frac{\underline{q}^2}{(z-\underline{q})^2}\left(1 + \frac{1}{\underline{q}}(z-\underline{q}) + \cdots\right).$$

Hence,

$$\frac{R_2}{(\tau-Z)^2} + \frac{R_1}{\tau - Z} + \cdots = -\frac{R_2\pi^2}{(\log z - \log \underline{q})^2} + \frac{R_1 \pi i}{\log z - \log \underline{q}} + \cdots$$

$$
\begin{aligned}
&= -\frac{R_2\pi^2\underline{q}^2}{(z-\underline{q})^2}\left(1+\frac{1}{\underline{q}}(z-\underline{q})+\cdots\right)\\
&\quad+\frac{R_1\pi i\underline{q}}{z-\underline{q}}\left(1+\frac{1}{2\underline{q}}(z-\underline{q})+\cdots\right)+\cdots\\
&= -\frac{R_2\pi^2\underline{q}^2}{(z-\underline{q})^2}+\frac{1}{z-\underline{q}}\left(-R_2\pi^2\underline{q}+R_1\pi i\underline{q}\right)+\cdots .
\end{aligned}
\tag{11.10.21}
$$

Therefore, by (11.10.20), (11.10.19), and (11.10.21),

$$
\operatorname{Res}\left(\frac{f(z)}{z-q},\underline{q}\right)=\frac{1}{\underline{q}-q}\left(-R_2\pi^2\underline{q}+R_1\pi i\underline{q}\right)+\frac{R_2\pi^2\underline{q}^2}{(\underline{q}-q)^2}. \tag{11.10.22}
$$

By a similar calculation,

$$
\operatorname{Res}\left(\frac{f(z)}{z-q},-\underline{q}\right)=\frac{1}{-\underline{q}-q}\left(R_2\pi^2\underline{q}-R_1\pi i\underline{q}\right)+\frac{R_2\pi^2\underline{q}^2}{(-\underline{q}-q)^2}. \tag{11.10.23}
$$

Hence, by (11.10.18), (11.10.22), (11.10.23), (11.10.16), and (11.10.17),

$$
\begin{aligned}
f(q) &= -\sum_{c,d}\left(\left(\frac{-\pi^2\underline{q}r_2}{(c\alpha+d)^{n+4}}-\frac{\pi i r_2 c(n+2)\underline{q}}{(c\alpha+d)^{n+3}}+\frac{\pi i r_1\underline{q}}{(c\alpha+d)^{n+2}}\right)\frac{1}{\underline{q}-q}\right.\\
&\quad+\frac{r_2\pi^2\underline{q}^2}{(c\alpha+d)^{n+4}(\underline{q}-q)^2}\\
&\quad+\left(\frac{\pi^2\underline{q}r_2}{(c\alpha+d)^{n+4}}+\frac{\pi i r_2 c(n+2)\underline{q}}{(c\alpha+d)^{n+3}}-\frac{\pi i r_1\underline{q}}{(c\alpha+d)^{n+2}}\right)\frac{1}{-\underline{q}-q}\\
&\qquad\left.+\frac{r_2\pi^2\underline{q}^2}{(c\alpha+d)^{n+4}(\underline{q}+q)^2}\right)\\
&= 2\pi i\sum_{c,d}\left\{\frac{cr_2(n+2)}{(c\alpha+d)^{n+3}}-\frac{r_1}{(c\alpha+d)^{n+2}}\right\}\frac{1}{1-(q/\underline{q})^2}\\
&\quad+2\pi^2 r_2\sum_{c,d}\frac{1}{(c\alpha+d)^{n+4}}\frac{1}{1-(q/\underline{q})^2}\\
&\quad-2\pi^2 r_2\sum_{c,d}\frac{1}{(c\alpha+d)^{n+4}}\frac{1+(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2}\\
&= 2\pi i\sum_{c,d}\left\{\frac{cr_2(n+2)}{(c\alpha+d)^{n+3}}-\frac{r_1}{(c\alpha+d)^{n+2}}\right\}\frac{1}{1-(q/\underline{q})^2}\\
&\quad+2\pi^2 r_2\sum_{c,d}\frac{1}{(c\alpha+d)^{n+4}}\left\{\frac{1}{1-(q/\underline{q})^2}-\frac{1+(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= 2\pi i \sum_{c,d} \left\{ \frac{cr_2(n+2)}{(c\alpha+d)^{n+3}} - \frac{r_1}{(c\alpha+d)^{n+2}} \right\} \frac{1}{1-(q/\underline{q})^2} \\
&\quad - 4\pi^2 r_2 \sum_{c,d} \frac{1}{(c\alpha+d)^{n+4}} \frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2},
\end{aligned}
$$

where the sum on c, d is as stated in Entry 11.10.1. This proves (11.10.3).

We next prove (11.10.6). From (11.10.2), since $q = e^{\pi i \tau}$,

$$
\begin{aligned}
f(q) &= \frac{\ell_2}{e^{2\pi i\alpha}(e^{\pi i(\tau-\alpha)}-1)^2} + \frac{\ell_1}{e^{\pi i\alpha}(e^{\pi i(\tau-\alpha)}-1)} + \cdots \\
&= \frac{\ell_2}{e^{2\pi i\alpha}(\pi i(\tau-\alpha)+\frac{1}{2}(\pi i)^2(\tau-\alpha)^2+\cdots)^2} + \frac{\ell_1}{e^{\pi i\alpha}(\pi i(\tau-\alpha)+\cdots)} + \cdots \\
&= -\frac{\ell_2}{e^{2\pi i\alpha}\pi^2(\tau-\alpha)^2(1+\pi i(\tau-\alpha)+\cdots)} + \frac{\ell_1}{e^{\pi i\alpha}(\pi i(\tau-\alpha))} + \cdots \\
&= -\frac{\ell_2}{e^{2\pi i\alpha}\pi^2(\tau-\alpha)^2} - \frac{\ell_2}{e^{2\pi i\alpha}\pi i(\tau-\alpha)} + \frac{\ell_1}{e^{\pi i\alpha}\pi i(\tau-\alpha)} + \cdots . \qquad (11.10.24)
\end{aligned}
$$

If we now compare the far right side of (11.10.24) with the right side of (11.10.2), we deduce (11.10.6). □

Lemma 11.10.1. *As in the general setting* (11.10.2), *put*

$$
\frac{1}{B^2(q)} = \frac{\ell_2}{(q-\underline{q})^2} + \frac{\ell_1}{q-\underline{q}} + \cdots, \qquad (11.10.25)
$$

where $\underline{q} = e^{\pi i\alpha}$, and where now $\alpha = 1+i$. Then

$$
\ell_1 = -\frac{B''(\underline{q})}{(B'(\underline{q}))^3} \qquad \text{and} \qquad \ell_2 = \frac{1}{(B'(\underline{q}))^2}. \qquad (11.10.26)
$$

Proof. Since $B(\underline{q}) = 0$,

$$
\begin{aligned}
\frac{1}{B^2(q)} &= \frac{1}{\left\{B'(\underline{q})(q-\underline{q}) + \frac{1}{2}B''(\underline{q})(q-\underline{q})^2 + \cdots\right\}^2} \\
&= \left\{ \frac{1}{B'(\underline{q})(q-\underline{q})} - \frac{B''(\underline{q})}{2B'(\underline{q})^2} + \cdots \right\}^2 \\
&= \frac{1}{B'(\underline{q})^2(q-\underline{q})^2} - \frac{f''(\underline{q})}{B'(\underline{q})^3(q-\underline{q})} + \cdots . \qquad (11.10.27)
\end{aligned}
$$

The values (11.10.26) now follow from (11.10.25) and (11.10.27). □

The coefficients of $1/B^2(q)$ are closely related to those for $1/R(q)$, which were established in Hardy and Ramanujan's paper [177, Theorem 3], [242, p. 319].

Entry 11.10.1 (pp. 97, 105, 114, 119, 123). *Define the coefficients p_n by*

$$\frac{1}{R(q^2)} =: \sum_{n=0}^{\infty} p_n q^{2n}, \qquad |q| < q_0. \tag{11.10.28}$$

Then, for $n \geq 0$,

$$p_n = \sum_{(\mu)} T_\mu(n), \tag{11.10.29}$$

where μ runs over all integers of the form (11.4.4), *and where*

$$T_1(n) = \frac{2}{Q^2(e^{-2\pi})} e^{2n\pi}, \tag{11.10.30}$$

$$T_2(n) = \frac{2}{Q^2(e^{-2\pi})} \frac{(-1)^n}{2^4} e^{n\pi}, \tag{11.10.31}$$

and, for $\mu > 2$,

$$T_\mu(n) = \frac{2}{Q^2(e^{-2\pi})} \frac{e^{2n\pi/\mu}}{\mu^4} \sum_{c,d} 2\cos\left((ac+bd)\frac{2\pi n}{\mu} + 8\tan^{-1}\frac{c}{d}\right), \tag{11.10.32}$$

where the sum is over all pairs (c, d), where (c, d) is a distinct solution to $\mu = c^2 + d^2$ and (a, b) is any solution to $ad - bc = 1$. Also, distinct solutions (c, d) to $\mu = c^2 + d^2$ give rise to distinct terms in the sum in (11.10.32).

We are now ready to state Ramanujan's theorem on the coefficients of $1/B^2(q)$.

Entry 11.10.2 (p. 119). *Define the coefficients b'_n by*

$$\frac{1}{B^2(q^2)} =: \sum_{n=0}^{\infty} b'_n q^{2n}, \qquad |q| < q_0.$$

Then,

$$b'_n = 18 \sum_{(\mu_e)} \left(n + \frac{3\mu_e}{2\pi}\right) T_{\mu_e}(n), \tag{11.10.33}$$

where the sum is over all even integers μ of the form (11.4.4), *and where $T_{\mu_e}(n)$ is defined by* (11.10.30), (11.10.31), *and* (11.10.32).

Proof. Throughout the proof we frequently and tacitly use the equalities

$$P(-e^{-\pi}) = 2P(e^{-2\pi}), \quad Q(-e^{-\pi}) = -4Q(e^{-2\pi}), \quad R(-e^{-\pi}) = R(e^{-2\pi}) = 0,$$

where the first equality follows from (11.7.4) and the equality $B(-e^{-\pi}) = 0$; the second comes from Entry 11.7.1 (iii) (or (11.8.16)); and the third arises from Entry 11.7.1 (ii), the equality $B(-e^{-\pi}) = 0$, and the fact that $e^{-2\pi}$ is a zero of $R(q)$ [246, p. 198].

By (11.7.4) and (11.1.7),

$$\begin{aligned}B'(-e^{-\pi}) &= \frac{d}{dq}\left(2P(q^2) - P(q)\right)\Big|_{q=-e^{-\pi}} \\ &= \frac{P^2(q^2)-Q(q^2)}{3q} - \frac{P^2(q)-Q(q)}{12q}\Big|_{q=-e^{-\pi}} = \frac{2Q(e^{-2\pi})}{3e^{-\pi}}.\end{aligned} \tag{11.10.34}$$

Next, by (11.10.34), (11.1.7), (11.1.8), and (11.8.10),

$$\begin{aligned}B''(-e^{-\pi}) &= \frac{1}{12q^2}\Bigg\{\Bigg(16qP(q^2)\frac{P^2(q^2)-Q(q^2)}{12q^2} - 8q\frac{P(q^2)Q(q^2)-R(q^2)}{3q^2} \\ &\qquad -2P(q)\frac{P^2(q)-Q(q)}{12q} + \frac{P(q)Q(q)-R(q)}{3q}\Bigg)q \\ &\qquad - 4P^2(q^2) + 4Q(q^2) + P(q^2) - Q(q)\Bigg\}\Bigg|_{q=-e^{-\pi}} \\ &= \frac{1}{12q^2}\left\{-4P(q^2)Q(q^2) + \frac{P(q)Q(q)}{2} + 8Q(q^2)\right\}\Bigg|_{q=-e^{-\pi}} \\ &= \frac{1}{12e^{-2\pi}}\left\{-8P(e^{-2\pi})Q(e^{-2\pi}) + 8Q(e^{-2\pi})\right\} \\ &= \frac{2Q(e^{-2\pi})}{3e^{-2\pi}}\left(1-\frac{3}{\pi}\right).\end{aligned} \tag{11.10.35}$$

By the chain rule and (11.10.34) and (11.10.35), respectively, it follows that

$$\frac{B(q^2)}{dq}\Bigg|_{q=ie^{-\pi/2}} = 2qB'(q^2)\big|_{q=ie^{-\pi/2}} = \frac{4iQ(e^{-2\pi})}{3e^{-\pi/2}}$$

and

$$\begin{aligned}\frac{d^2B(q^2)}{dq^2}\Bigg|_{q=ie^{-\pi/2}} &= 2B'(q^2) + 4q^2B''(q^2)\big|_{q=ie^{-\pi/2}} \\ &= \frac{4Q(e^{-2\pi})}{3e^{-\pi}} - \frac{8e^{-\pi}Q(e^{-2\pi})}{3e^{-2\pi}}\left(1-\frac{3}{\pi}\right) \\ &= \frac{4Q(e^{-2\pi})}{3e^{-\pi}}\left(\frac{6}{\pi}-1\right).\end{aligned}$$

It follows from (11.10.26) that

$$\ell_2 = -\frac{9e^{-\pi}}{16Q^2(e^{-2\pi})}$$

and

$$\ell_1 = -\frac{27ie^{-3\pi/2}}{64Q^3(e^{-2\pi})}\frac{4Q(e^{-2\pi})}{3e^{-\pi}}\left(\frac{6}{\pi}-1\right) = -\frac{9ie^{-\pi/2}}{16Q^2(e^{-2\pi})}\left(\frac{6}{\pi}-1\right).$$

Using the calculations above in (11.10.6), we further find that

$$r_2 = -\frac{9}{16\pi^2Q^2(e^{-2\pi})} \tag{11.10.36}$$

and

$$r_1 = -\frac{i}{\pi e^{-\pi}}\left(-\frac{9e^{-\pi}}{16Q^2(e^{-2\pi})}\right) + \frac{9i}{16\pi Q^2(e^{-2\pi})}\left(\frac{6}{\pi}-1\right) = \frac{27i}{8\pi^2Q^2(e^{-2\pi})}. \tag{11.10.37}$$

We now apply Theorem 11.10.1 to $1/B^2(q^2)$. Note that $n=4$ and that $\underline{q}$ is defined by (11.8.13). Accordingly,

$$\begin{aligned}\frac{1}{B^2(q^2)} &= 2\pi i \sum_{\substack{(c,d)\\ c \text{ even}}} \left\{\frac{6cr_2}{(c\frac{1+i}{2}+d)^7} - \frac{r_1}{(c\frac{1+i}{2}+d)^6}\right\}\frac{1}{1-(q/\underline{q})^2}\\ &\quad - 4\pi^2 r_2 \sum_{\substack{(c,d)\\ c \text{ even}}} \frac{1}{(c\frac{1+i}{2}+d)^8}\frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2}.\end{aligned}$$

We use the calculations (11.8.21) and (11.8.22) with $c'=c+d$. Since c is even and d is odd, then c' is odd and $\mu = c'^2+d^2$ is even. Replacing c' by c, we find that

$$\begin{aligned}\frac{1}{B^2(q^2)} &= 2\pi i \sum_{\substack{(c,d)\\ c^2+d^2 \text{ even}}} \left\{\frac{(1+i)^7 6(c-d)r_2}{(ci+d)^7} - \frac{(1+i)^6 r_1}{(ci+d)^6}\right\}\frac{1}{1-(q/\underline{q})^2}\\ &\quad - 4\pi^2 r_2 \sum_{\substack{(c,d)\\ c^2+d^2 \text{ even}}} \frac{(1+i)^8}{(ci+d)^8}\frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2}.\end{aligned}$$

Using (11.10.36) and (11.10.37), we deduce that

$$\begin{aligned}&\frac{1}{B^2(q^2)}\\ &\quad = 2\pi i \sum_{\substack{(c,d)\\ c^2+d^2 \text{ even}}} \left\{\frac{-8i}{(ci+d)^7}\left(-\frac{6\cdot 9(c-d)(1+i)}{16\pi^2Q^2(e^{-2\pi})} - \frac{27i(ci+d)}{8\pi^2Q^2(e^{-2\pi})}\right)\right\}\\ &\qquad \times \frac{1}{1-(q/\underline{q})^2} + \frac{36}{Q^2(e^{-2\pi})} \sum_{\substack{(c,d)\\ c^2+d^2 \text{ even}}} \frac{1}{(ci+d)^8}\frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2}\\ &\quad = \frac{16}{\pi Q^2(e^{-2\pi})} \sum_{\substack{(c,d)\\ c^2+d^2 \text{ even}}} \frac{1}{(ci+d)^7}\left(-\frac{27(ci-d)}{8}\right)\frac{1}{1-(q/\underline{q})^2}\end{aligned}$$

$$+\frac{36}{Q^2(e^{-2\pi})}\sum_{\substack{(c,d)\\ c^2+d^2\ \text{even}}}\frac{1}{(ci+d)^8}\frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2}$$

$$=-\frac{54}{\pi Q^2(e^{-2\pi})}\sum_{\substack{(c,d)\\ c^2+d^2\ \text{even}}}\frac{(ci-d)(ci+d)}{(ci+d)^8}\frac{1}{1-(q/\underline{q})^2}$$

$$+\frac{36}{Q^2(e^{-2\pi})}\sum_{\substack{(c,d)\\ c^2+d^2\ \text{even}}}\frac{1}{(ci+d)^8}\frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2}$$

$$=\frac{54}{\pi Q^2(e^{-2\pi})}\sum_{\substack{(c,d)\\ c^2+d^2\ \text{even}}}\frac{\mu}{(ci+d)^8}\frac{1}{1-(q/\underline{q})^2}$$

$$+\frac{36}{Q^2(e^{-2\pi})}\sum_{\substack{(c,d)\\ c^2+d^2\ \text{even}}}\frac{1}{(ci+d)^8}\frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2}.$$

Hence, as in the proof in [177, Theorem 3], [242, p. 319] or Entry 11.4.1, we separate the terms for positive and negative c and observe that if c is replaced by $-c$ in $\underline{q}$, then $\underline{q}$ is replaced by $\overline{\underline{q}}$. From above, we then deduce that

$$\begin{aligned}
\frac{1}{B^2(q^2)} &= \frac{54}{\pi Q^2(e^{-2\pi})}\left(\frac{1}{2^4}\frac{1}{1+e^{\pi}q^2}\right.\\
&\quad\left.+\sum_{\substack{(c,d)\\ \mu>2}}\left\{\frac{\mu}{(ci+d)^8}\frac{1}{1-(q/\underline{q})^2}+\frac{\mu}{(-ci+d)^8}\frac{1}{1-(q/\overline{\underline{q}})^2}\right\}\right)\\
&\quad+\frac{36}{Q^2(e^{-2\pi})}\left(\frac{1}{2^4}\frac{-e^{\pi}q^2}{1+e^{\pi}q^2}\right.\\
&\quad\left.+\sum_{\substack{(c,d)\\ \mu>2}}\left\{\frac{1}{(ci+d)^8}\frac{(q/\underline{q})^2}{(1-(q/\underline{q})^2)^2}+\frac{1}{(-ci+d)^8}\frac{(q/\overline{\underline{q}})^2}{(1-(q/\overline{\underline{q}})^2)^2}\right\}\right)\\
&= \frac{54}{\pi Q^2(e^{-2\pi})}\left(\frac{1}{2^4}\sum_{n=0}^{\infty}(-1)^n e^{\pi n}q^{2n}\right.\\
&\quad\left.+\sum_{\substack{(c,d)\\ \mu>2}}\left\{\frac{\mu}{(ci+d)^8}\sum_{n=0}^{\infty}\underline{q}^{-2n}q^{2n}+\frac{\mu}{(-ci+d)^8}\sum_{n=0}^{\infty}\overline{\underline{q}}^{-2n}q^{2n}\right\}\right)\\
&\quad+\frac{36}{Q^2(e^{-2\pi})}\left(\frac{1}{2^4}\sum_{n=0}^{\infty}(-1)^n n e^{\pi n}q^{2n}\right.
\end{aligned}$$

$$+\sum_{\substack{(c,d)\\ \mu>2}}\left\{\frac{1}{(ci+d)^8}\sum_{n=0}^{\infty} n\underline{q}^{-2n}q^{2n}+\frac{1}{(-ci+d)^8}\sum_{n=0}^{\infty} n\underline{\overline{q}}^{-2n}q^{2n}\right\}\Bigg).$$

Equating coefficients of $q^{2n}, n \geq 0$, on both sides and proceeding as in the proof in [177, Theorem 3], [242, p. 319] or Entry 11.4.1, we find that

$$\begin{aligned}
b'_n &= \frac{54}{\pi Q^2(e^{-2\pi})}\left(\frac{(-1)^n}{2^4}e^{\pi n}+\sum_{\substack{(\mu_e)\\ \mu_e>2}}\sum_{(c,d)}\frac{\mu_e e^{2\pi n/\mu_e}}{\mu_e^4}\right.\\
&\qquad\qquad\left.\times 2\cos\left(\frac{2\pi n}{\mu_e}(ac+bd)+8\tan^{-1}\frac{c}{d}\right)\right)\\
&\quad+\frac{36}{Q^2(e^{-2\pi})}\left(\frac{(-1)^n}{2^4}ne^{\pi n}+\sum_{\substack{(\mu_e)\\ \mu_e>2}}\sum_{(c,d)}\frac{ne^{2\pi n/\mu_e}}{\mu_e^4}\right.\\
&\qquad\qquad\left.\times \cos\left(\frac{2\pi n}{\mu_e}(ac+bd)+8\tan^{-1}\frac{c}{d}\right)\right)\\
&=\frac{27}{\pi}\sum_{(\mu_e)}\mu_e T_{\mu_e}(n)+18n\sum_{(\mu_e)}\mu_e T_{\mu_e}(n)\\
&=18\sum_{(\mu_e)}\left(\frac{3\mu_e}{2\pi}+n\right)T_{\mu_e}(n),
\end{aligned}$$

where the sums on μ_e and c, d are as given in the statement of Theorem 11.10.2. □

Using *Mathematica*, we calculated $b'_n, 1 \leq n \leq 10$, and the first two terms in (11.10.33). As with b_n, the accuracy is remarkable:

n	b'_n	$18\left((n+\frac{3}{\pi})T_2(n)+(n+\frac{15}{\pi})T_{10}(n)\right)$
1	-48	-48.001187
2	$1,680$	$1,679.997897$
3	$-52,032$	$-52,031.997988$
4	$1,508,496$	$1,508,496.002778$
5	$-41,952,672$	$-41,952,671.998915$
6	$1,133,840,832$	$1,133,840,831.996875$
7	$-30,010,418,304$	$-30,010,418,304.008563$
8	$781,761,426,576$	$781,761,426,576.003783$
9	$-20,110,673,188,848$	$-20,110,673,188,847.986981$
10	$512,123,093,263,584$	$512,123,093,263,584.006307$

Ramanujan's theorem on the coefficients of $1/B^2(q)$ is also closely related to the power series expansion in Entry 11.5.1. We restate that theorem here using the notation (11.10.30)–(11.10.32).

Entry 11.10.3 (p. 102). *Define the coefficients* η_n *by*

$$\frac{P(q^2)}{R(q^2)} =: \sum_{n=0}^{\infty} \eta_n q^{2n}, \qquad |q| < q_0. \tag{11.10.38}$$

Then, for $n \geq 0$,

$$\eta_n = \frac{3}{\pi} \sum_{(\mu)} \mu T_\mu(n), \tag{11.10.39}$$

where μ *runs over all integers of the form* (11.4.4), *and where* $T_\mu(n)$ *is defined by* (11.10.30)–(11.10.32).

Now observe from (11.10.29) and (11.10.38) that

$$\sum_{n=0}^{\infty} c_n q^n := q\left(\frac{1}{R(q)}\right)' + \frac{P(q)}{2R(q)} = \sum_{n=0}^{\infty} n p_n q^n + \frac{1}{2}\sum_{n=0}^{\infty} \eta_n q^n, \tag{11.10.40}$$

and so by (11.10.29) and (11.10.39),

$$c_n = \sum_{\mu} \left(n + \frac{3\mu}{2\pi}\right) T_\mu(n) =: c_{n,e} + c_{n,o}, \tag{11.10.41}$$

where $c_{n,e}$ and $c_{n,o}$ are the subseries over the even and odd values of μ, respectively. Equality (11.10.41) should be compared with (11.10.33); in particular, note that $b_n' = 18c_{n,e}$. Moreover, by (11.1.9),

$$\begin{aligned} q\left(\frac{1}{R(q)}\right)' + \frac{P(q)}{2R(q)} &= \frac{-P(q)R(q) + Q^2(q)}{2R^2(q)} + \frac{P(q)}{2R(q)} \\ &= \frac{1}{2}\left(\frac{Q(q)}{R(q)}\right)^2 =: \frac{1}{2}\sum_{n=0}^{\infty} d_n q^n. \end{aligned} \tag{11.10.42}$$

Defining $d_{n,e}$ and $d_{n,o}$ as we did above for $c_{n,e}$ and $c_{n,o}$, we see by (11.10.33) and (11.10.40)–(11.10.42) that

$$d_{n,e} = 2c_{n,e}, \qquad d_{n,o} = 2c_{n,o}, \qquad \text{and} \qquad b_n' = 9d_{n,e}.$$

Using the formula for the coefficients of $Q(q)/R(q)$ given in Entry 11.4.1, we can obtain a relation between these coefficients and the coefficients d_n above.

Although we have stated Theorem 11.10.1 only for modular forms on $\Gamma_0(2)$ with a single double pole on a fundamental region, there is an obvious analogue for modular forms on the full modular group. In fact, as a check on our work, we applied this analogue to $Q^2(q^2)/R^2(q^2)$ to show that $d_n = 2c_n$, where c_n is given by (11.10.41). Also, Lehner's theorem [201] can also now be obviously extended for forms with double poles.

11.11 A Calculation from [176]

On page 104 in his lost notebook [244], Ramanujan offers a calculation from [176] to illustrate the accuracy of their formula for the coefficients of $1/R(q)$ by taking only a small number of terms from their formula. Define the coefficients p_n, $n \geq 0$, by

$$\sum_{n=0}^{\infty} p_n q^n = \frac{1}{R(q)}, \qquad |q| < 1.$$

Ramanujan records the first thirteen coefficients. These coefficients are also recorded by Hardy and Ramanujan in their paper [176], [242, p. 317].

Entry 11.11.1 (p. 104).

$$\begin{aligned}
p_0 &= 1\\
p_1 &= 504\\
p_2 &= 270648\\
p_3 &= 144912096\\
p_4 &= 77599626552\\
p_5 &= 41553943041744\\
p_6 &= 22251789971649504\\
p_7 &= 11915647845248387520\\
p_8 &= 6380729991419236488504\\
p_9 &= 3416827666558895485479576\\
p_{10} &= 1829682703808504464920468048\\
p_{11} &= 979779820147442370107345764512\\
p_{12} &= 524663917940510191509934144603104
\end{aligned}$$

To calculate the coefficient p_{12}, Ramanujan takes six terms from their formula (11.10.29) of Entry 11.10.1; these terms are numerically equal to, respectively,

$$\begin{array}{r}
524663917940510190119197271938395.329\\
+1390736872662028.140\\
+2680.418\\
+0.130\\
-0.014\\
-0.003\\
\hline
524663917940510191509934144603104.000
\end{array}$$

For more details, see Hardy and Ramanujan's paper [176], [242, pp. 317, 320].

12

Two Letters on Eisenstein Series Written from Matlock House

12.1 Introduction

As we mentioned in Chapter 11, in their last joint paper, G.H. Hardy and Ramanujan [177], [242, pp. 310–321] established the following remarkable formula for the coefficients of $1/R(q)$.

Entry 12.1.1 (pp. 97, 105, 114, 119, 123). *Let*

$$\frac{1}{R(q^2)} = \sum_{n=0}^{\infty} \gamma_n q^{2n}, \qquad |q| < q_0.$$

Then,

$$\gamma_n = \frac{2}{C} \sum_{(\mu)} \frac{W_\mu(n)}{\mu^4} e^{2n\pi/\mu}, \tag{12.1.1}$$

where C is a constant defined in (11.5.3), *μ runs over the integers of the form* (11.4.4), *and $W_\mu(n)$ is defined in* (11.5.5) *and* (11.5.7). *Also, distinct solutions (c, d) to $\mu = c^2 + d^2$ give rise to distinct terms in the sum stated in* (12.1.1).

In another letter to Hardy from the English sanitarium Matlock House, Ramanujan derives upper and lower bounds for the number of terms in the formula (12.1.1) required to determine the value of each coefficient, that is, the number of terms required to produce an approximation that has the actual coefficient as the nearest integer. This letter was published with Ramanujan's "lost notebook" [23, pp. 97–101]. We will present an expanded version with more details of Ramanujan's argument in this chapter. In this letter, Ramanujan writes [244, p. 97], "In one of my letters I wrote about the least number of terms which will give the nearest integer to the actual coefficient in $1/g_3$ problem. It will be extremely difficult to prove such a result. But we can prove this much as follows." It is uncertain whether the letter to which Ramanujan

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refers still exists. However, pages 123–126 of [244] appear to be a portion of this letter. The beginning of the letter is clearly missing, and the editors label these pages sheets “from the LOST NOTEBOOK.”

12.2 A Lower Bound

Let

$$\ell(n) := \left[\frac{\frac{3}{4}n(1-\epsilon)}{\log^{3/2} n \sqrt{\prod_{p\equiv 1 \pmod 4} \left(1-\frac{1}{p^2}\right)}} \right], \tag{12.2.1}$$

where $[x]$ denotes the greatest integer less than or equal to x, $0 \leq \epsilon \leq 1$, and p runs over the primes congruent to 1 modulo 4.

Let $\sum_n$ be the sum on the right-hand side of (12.1.1), and let $\sum_n^{\ell(n)}$ be the sum of the first $\ell(n)$ terms of $\sum_n$, where all of the summands corresponding to one value of μ together count as one “term.”

Entry 12.2.1 (pp. 97–101). (*A Lower Bound for the Number of Terms Required*) *As n tends to ∞, there are terms in $\sum_n$ that are not in $\sum_n^{\ell(n)}$ and are arbitrarily large.*

Proof. Strictly speaking, this theorem does not imply that there is a large gap between $\sum_n^{\ell(n)}$ and γ_n. Rather, it indicates that we cannot have confidence in $\sum_n^{\ell(n)}$ as an approximation to γ_n because subsequent terms in the series $\sum_n$ are quite large.

We define $c(m)$ to be the number of integers less than or equal to m that can be represented as the sum of two coprime squares, i.e., that can be represented in the form (11.4.4). We obtain an asymptotic formula for $c(m)$.

We define $r(m)$ to be the number of integers less than or equal to m that can be represented as the sum of two squares. Such integers are of the form [175, p. 299]

$$t = 2^a \prod_{i=1}^{r} p_i^{a_i} \prod_{j=i}^{s} q_j^{2b_j}, \tag{12.2.2}$$

where a is a nonnegative integer, p_i is a prime of the form $4k+1$, q_j is a prime of the form $4k+3$, and a_i and b_j are positive integers, $1 \leq i \leq r$, $1 \leq j \leq s$.

From (11.4.4) and (12.2.2), we see that the integers that are counted in $r(m)$ but are not counted in $c(m)$ are those that are divisible by 4 or by the square of a prime of the form $4k+3$. Suppose we choose one of these integers that is divisible by 4. Then it must be of the form

$$4\left(2^a\prod_{i=1}^{r}p_i^{a_i}\prod_{j=1}^{s}q_j^{2b_j}\right),$$

and therefore, there are $r(m/4)$ such integers less than or equal to m. Similarly, there are $r(m/9)$ integers divisible by 9 that are counted in $r(m)$ but not in $c(m)$. Recalling that the integers counted in $r(m)$ but not in $c(m)$ are those that are divisible by 4 or the square of a prime of the form $4k+3$, we use an inclusion–exclusion argument to deduce that

$$\begin{aligned} c(m) &= r(m) - r\left(\frac{m}{4}\right) - r\left(\frac{m}{9}\right) + r\left(\frac{m}{36}\right) - r\left(\frac{m}{49}\right) - \cdots \\ &= \sum_{(g)} r\left(\frac{m}{g}\right)(-1)^{w(g)}, \end{aligned} \tag{12.2.3}$$

where $w(g)$ is the number of prime factors of g, not including multiplicities, and g runs over integers of the form

$$g := 2^{2c}\prod_{j=1}^{N}q_j^{2c_j}, \tag{12.2.4}$$

where c and c_j are 0 or 1.

E. Landau [198, p. 66] and Ramanujan [56, pp. 60–66] both showed that

$$r(m) = \frac{Bm}{\sqrt{\log m}} + O\left(\frac{m}{\log^{3/2} m}\right), \tag{12.2.5}$$

where

$$B = \frac{1}{\sqrt{2\displaystyle\prod_{p\equiv 3 \pmod 4}\left(1-\frac{1}{p^2}\right)}}. \tag{12.2.6}$$

We use this formula for $r(m)$ to obtain a similar formula for $c(m)$.

Lemma 12.2.1. *We have*

$$c(m) = \frac{3}{2\pi}\frac{1}{\sqrt{\displaystyle\prod_{p\equiv 1 \pmod 4}\left(1-\frac{1}{p^2}\right)}}\frac{m}{\sqrt{\log m}} + O\left(\frac{m}{\log^{3/2} m}\right).$$

Proof. Throughout the proof, g denotes an integer of the form (12.2.4). Note that we can rewrite (12.2.3) as

$$c(m) = \sum_{g\le m} r\left(\frac{m}{g}\right)(-1)^{w(g)}, \tag{12.2.7}$$

because $r\left(\frac{m}{g}\right) = 0$ for $g > m$. We write (12.2.7) as two sums, namely,

$$c(m) = \sum_{g<m^{2/3}} r\left(\frac{m}{g}\right)(-1)^{w(g)} \quad + \sum_{m^{2/3}\le g\le m} r\left(\frac{m}{g}\right)(-1)^{w(g)} . \tag{12.2.8}$$

We estimate each sum separately. For $m^{2/3} \le g \le m$,

$$r\left(\frac{m}{g}\right) \le r\left(\frac{m}{m^{2/3}}\right) = r\left(m^{1/3}\right) \le m^{1/3} . \tag{12.2.9}$$

Since g runs over integers of the form (12.2.4) and these are all squares, the number of terms in the second sum in (12.2.8) is less than or equal to $\sqrt{m}$. From this and (12.2.9), we see that the second sum in (12.2.8) is $O(m^{5/6})$. Thus from (12.2.8), it follows that

$$c(m) = \sum_{g<m^{2/3}} r\left(\frac{m}{g}\right)(-1)^{w(g)} + O(m^{5/6}).$$

Thus, by (12.2.5),

$$c(m) = \sum_{g<m^{2/3}} \left\{ \frac{Bm/g}{\sqrt{\log(m/g)}}(-1)^{w(g)} + O\left(\frac{m/g}{\log^{3/2}(m/g)}\right)\right\} + O(m^{5/6}) . \tag{12.2.10}$$

We simplify the sum involving the O-term. Note that for $g < m^{2/3}$,

$$\begin{aligned}
\sum_{g<m^{2/3}} \frac{m/g}{\log^{3/2}(m/g)} &= \sum_{g<m^{2/3}} \frac{m/g}{\log^{3/2} m(1-\log g/\log m)^{3/2}} \\
&< \sum_{g<m^{2/3}} \frac{m/g}{\log^{3/2} m(1-\log m^{2/3}/\log m)^{3/2}} \\
&= \sum_{g<m^{2/3}} \frac{m/g}{\log^{3/2} m(1-2/3)^{3/2}} \\
&= O\left(\frac{m}{\log^{3/2} m} \sum_{g<m^{2/3}} \frac{1}{g}\right) \\
&= O\left(\frac{m}{\log^{3/2} m} \sum_{n=1}^{\infty} \frac{1}{n^2}\right) \\
&= O\left(\frac{m}{\log^{3/2} m}\right),
\end{aligned}$$

where in the penultimate step we used the fact that g runs over a subset of the squares. So by (12.2.10),

$$c(m) = \sum_{g<m^{2/3}} \left\{ \left(\frac{Bm/g}{\sqrt{\log(m/g)}} \right) (-1)^{w(g)} \right\} + O\left(\frac{m}{\log^{3/2} m} \right).$$

Now we express the sum above in terms of $\log m$ rather than $\log(m/g)$. Using the fact that for $g < m^{2/3}$,

$$\begin{aligned} \frac{1}{\sqrt{\log(m/g)}} &= \frac{1}{\sqrt{\log m}} \frac{1}{\sqrt{1 - \log g / \log m}} \\ &= \frac{1}{\sqrt{\log m}} \left(1 + O\left(\frac{\log g}{\log m} \right) \right) \\ &= \frac{1}{\sqrt{\log m}} + O\left(\frac{\log g}{\log^{3/2} m} \right), \end{aligned}$$

we find that

$$\begin{aligned} c(m) &= \sum_{g<m^{2/3}} \left\{ \frac{Bm}{g} (-1)^{\omega(g)} \left(\frac{1}{\sqrt{\log m}} + O\left(\frac{\log g}{\log^{3/2} m} \right) \right) \right\} + O\left(\frac{m}{\log^{3/2} m} \right) \\ &= \sum_{g<m^{2/3}} \frac{Bm/g}{\sqrt{\log m}} (-1)^{\omega(g)} + O\left(\frac{m}{\log^{3/2} m} \sum_{g<m^{2/3}} \frac{\log g}{g} \right) + O\left(\frac{m}{\log^{3/2} m} \right) \\ &= \sum_{g<m^{2/3}} \frac{Bm/g}{\sqrt{\log m}} (-1)^{\omega(g)} + O\left(\frac{m}{\log^{3/2} m} \right), \end{aligned} \tag{12.2.11}$$

where in the second sum of the penultimate line, we used the fact that g runs over a subset of the squares.

We now show that (12.2.11) is true even if we remove the condition $g < m^{2/3}$ from the sum. Recall that g runs over squares, and so the sum in (12.2.11) is absolutely convergent. Hence,

$$\begin{aligned} c(m) = \sum_{g} \left\{ \frac{Bm/g}{\sqrt{\log m}} (-1)^{w(g)} \right\} - \sum_{g \geq m^{2/3}} \left\{ \frac{Bm/g}{\sqrt{\log m}} (-1)^{w(g)} \right\} \\ + O\left(\frac{m}{\log^{3/2} m} \right). \end{aligned}$$

However,

$$\begin{aligned} \left| \sum_{g \geq m^{2/3}} \left\{ \frac{Bm/g}{\sqrt{\log m}} (-1)^{w(g)} \right\} \right| &= \left| \frac{Bm}{\sqrt{\log m}} \sum_{g \geq m^{2/3}} \frac{(-1)^{w(g)}}{g} \right| \\ &< \frac{Bm}{\sqrt{\log m}} \sum_{g \geq m^{2/3}} \frac{1}{g} \\ &< \frac{Bm}{\sqrt{\log m}} \sum_{n \geq m^{1/3}}^{\infty} \frac{1}{n^2} \end{aligned}$$

$$
\begin{aligned}
&= O\left(\frac{m}{\sqrt{\log m}}\frac{1}{m^{1/3}}\right)\\
&= O\left(\frac{m^{2/3}}{\sqrt{\log m}}\right)\\
&= O\left(\frac{m}{\log^{3/2} m}\right), \qquad (12.2.12)
\end{aligned}
$$

and so

$$
c(m) = \frac{Bm}{\sqrt{\log m}} \sum_{(g)} \frac{(-1)^{w(g)}}{g} + O\left(\frac{m}{\log^{3/2} m}\right). \qquad (12.2.13)
$$

Recall that g runs over integers of the form (12.2.4), where the q_j are primes congruent to $3\,(\mathrm{mod}\,4)$. Thus, using the Euler product representation for the sum in (12.2.13), we find that

$$
c(m) = \frac{Bm}{\sqrt{\log m}}\left(1-\frac{1}{2^2}\right)\prod_{p\equiv 3\,(\mathrm{mod}\,4)}\left(1-\frac{1}{p^2}\right) + O\left(\frac{m}{\log^{3/2} m}\right).
$$

Using the definition of B in (12.2.6) with the representation above, we deduce that

$$
\begin{aligned}
c(m) &= \frac{1}{\sqrt{2\displaystyle\prod_{p\equiv 3\,(\mathrm{mod}\,4)}\left(1-\frac{1}{p^2}\right)}}\frac{m}{\sqrt{\log m}}\left(\frac{3}{4}\right)\prod_{p\equiv 3\,(\mathrm{mod}\,4)}\left(1-\frac{1}{p^2}\right)\\
&\quad + O\left(\frac{m}{\log^{3/2} m}\right)\\
&= \frac{3}{4\sqrt{2}}\sqrt{\prod_{p\equiv 3\,(\mathrm{mod}\,4)}\left(1-\frac{1}{p^2}\right)}\frac{m}{\sqrt{\log m}} + O\left(\frac{m}{\log^{3/2} m}\right).
\end{aligned}
$$

Using the elementary fact

$$
\prod_p\left(1-\frac{1}{p^2}\right) = \frac{1}{\zeta(2)} = \frac{6}{\pi^2},
$$

we obtain our desired result, namely,

$$
c(m) = \frac{3}{4\sqrt{2}}\sqrt{\frac{\displaystyle\prod_p\left(1-\frac{1}{p^2}\right)}{\left(1-\frac{1}{2^2}\right)\displaystyle\prod_{p\equiv 1\,(\mathrm{mod}\,4)}\left(1-\frac{1}{p^2}\right)}}\frac{m}{\sqrt{\log m}}
$$

$$
\begin{aligned}
&+ O\left(\frac{m}{\log^{3/2} m}\right) \\
&= \frac{3}{4\sqrt{2}} \sqrt{\frac{\dfrac{6}{\pi^2}}{\dfrac{3}{4} \displaystyle\prod_{p\equiv 1 \,(\mathrm{mod}\, 4)} \left(1 - \frac{1}{p^2}\right)}} \frac{m}{\sqrt{\log m}} \\
&+ \quad O\left(\frac{m}{\log^{3/2} m}\right) \\
&= \frac{3}{4\sqrt{2}} \sqrt{\frac{8}{\pi^2}} \frac{1}{\sqrt{\displaystyle\prod_{p\equiv 1 \,(\mathrm{mod}\, 4)} \left(1 - \frac{1}{p^2}\right)}} \frac{m}{\sqrt{\log m}} + O\left(\frac{m}{\log^{3/2} m}\right) \\
&= \frac{3}{2\pi} \frac{1}{\sqrt{\displaystyle\prod_{p\equiv 1 \,(\mathrm{mod}\, 4)} \left(1 - \frac{1}{p^2}\right)}} \frac{m}{\sqrt{\log m}} + O\left(\frac{m}{\log^{3/2} m}\right) .
\end{aligned}
$$

This concludes the proof of the lemma. □

In the sum $\sum_n^{\ell(n)}$, the index μ runs over the integers of the form (11.4.4), that is, the integers in the sequence 1, 2, 5, 10, 13 . . . . We denote the integers in this sequence by $\mu_1, \mu_2, \mu_3, \ldots$. Then the value of μ corresponding to the final term of the sum $\sum_n^{\ell(n)}$ is $\mu_{\ell(n)}$. Thus,

$$
c(\mu_{\ell(n)}) = \ell(n) . \tag{12.2.14}
$$

We now obtain an asymptotic formula for $\mu_{\ell(n)}$ in terms of n. Then we use this formula to obtain a lower bound for certain terms of $\sum_n$ that are not in $\sum_n^{\ell(n)}$ and in this way prove the theorem.

By (12.2.14), Lemma 12.2.1, and (12.2.1),

$$
\begin{aligned}
1 &= \lim_{n\to\infty} \frac{\ell(n)}{\dfrac{3}{2\pi} \dfrac{1}{\sqrt{\displaystyle\prod_{p\equiv 1 \,(\mathrm{mod}\, 4)} \left(1 - \frac{1}{p^2}\right)}} \dfrac{\mu_{\ell(n)}}{\sqrt{\log \mu_{\ell(n)}}}} \\
&= \lim_{n\to\infty} \frac{\left[\dfrac{\frac{3}{4} n(1-\epsilon)}{\log^{3/2} n \sqrt{\displaystyle\prod_{p\equiv 1 \,(\mathrm{mod}\, 4)}} \left(1 - \frac{1}{p^2}\right)}\right]}{\dfrac{3}{2\pi} \dfrac{1}{\sqrt{\displaystyle\prod_{p\equiv 1 \,(\mathrm{mod}\, 4)} \left(1 - \frac{1}{p^2}\right)}} \dfrac{\mu_{\ell(n)}}{\sqrt{\log \mu_{\ell(n)}}}}
\end{aligned}
$$

$$= \lim_{n\to\infty} \frac{\pi(1-\epsilon)n/\log^{3/2} n}{2\mu_{\ell(n)}/\sqrt{\log \mu_{\ell(n)}}} . \tag{12.2.15}$$

Note that if $a_n, b_n \to \infty$, then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

implies that

$$\lim_{n\to\infty} \frac{\log a_n}{\log b_n} = 1,$$

because

$$\begin{aligned} \lim_{n\to\infty} \frac{\log a_n}{\log b_n} &= \lim_{n\to\infty} \frac{\log a_n - \log b_n + \log b_n}{\log b_n} \\ &= \lim_{n\to\infty} \frac{\log(a_n/b_n)}{\log b_n} + 1 = 1 . \end{aligned}$$

Thus,

$$1 = \lim_{n\to\infty} \frac{\log \pi(1-\epsilon) + \log n - \log(\log n)^{3/2}}{\log 2 + \log \mu_{\ell(n)} - \log(\log \mu_{\ell(n)})^{1/2}} = \lim_{n\to\infty} \frac{\log n}{\log \mu_{\ell(n)}} . \tag{12.2.16}$$

By (12.2.15) and (12.2.16),

$$1 = \lim_{n\to\infty} \frac{\pi(1-\epsilon)}{2} \frac{n/(\log n)^{3/2}}{\mu_{\ell(n)}/\sqrt{\log n}} ,$$

so that

$$\mu_{\ell(n)} \sim \frac{\pi(1-\epsilon)}{2} \frac{n}{\log n} . \tag{12.2.17}$$

We let $\hat{p}_n$ denote the smallest prime of the form $4k+1$ that is greater than $\mu_{\ell(n)}$. Note that

$$\hat{p}_n \sim \mu_{\ell(n)} \tag{12.2.18}$$

as n tends to ∞, by Dirichlet's theorem for primes in arithmetic progressions.

Consider now the coefficients

$$\gamma_n, \gamma_{n-1}, \ldots \quad \text{and} \quad \gamma_{n-\hat{p}_n+1} .$$

Because $\hat{p}_n$ is of the form (11.4.4), the series $\sum_n$, $\sum_{n-1}, \ldots,$ and $\sum_{n-\hat{p}_n+1}$ for these coefficients will contain the terms

$$\frac{2}{C} \frac{2\cos\left\{2(ac+bd)\dfrac{n\pi}{\hat{p}_n} + 8\arctan\dfrac{c}{d}\right\} e^{2n\pi/\hat{p}_n}}{\hat{p}_n^4} ,$$

$$\frac{2}{C}\frac{2\cos\left\{2(ac+bd)\dfrac{(n-1)\pi}{\hat{p}_n}+8\arctan\dfrac{c}{d}\right\}e^{2(n-1)\pi/\hat{p}_n}}{\hat{p}_n^4},$$
$$\ldots, \tag{12.2.19}$$

and

$$\frac{2}{C}\frac{2\cos\left\{2(ac+bd)\dfrac{(n-\hat{p}_n+1)\pi}{\hat{p}_n^4}+8\arctan\dfrac{c}{d}\right\}e^{2(n-\hat{p}_n+1)\pi/\hat{p}_n}}{\hat{p}_n}, \tag{12.2.20}$$

respectively. Recall that in our definition of $\sum_n^{\ell(n)}$, we mentioned that all of the summands corresponding to one value of μ together count as one term, and recall that since $\hat{p}_n$ is a prime of the form $4k+1$, it has a unique representation as a sum of two squares. Thus, indeed the terms are of the forms (12.2.19) and (12.2.20). However, because

$$\hat{p}_n > \mu_{\ell(n)} > \mu_{\ell(n-1)} > \cdots > \mu_{\ell(n-\hat{p}_n+1)},$$

the truncated series

$$\sum\nolimits_n^{\ell(n)}, \quad \sum\nolimits_{n-1}^{\ell(n-1)}, \ldots, \sum\nolimits_{n-\hat{p}_n+1}^{\ell(n-\hat{p}_n+1)}$$

does not contain these terms. For n sufficiently large, at least one of the expressions

$$2\cos\left\{2(ac+bd)\frac{n\pi}{\hat{p}_n}+8\arctan\frac{c}{d}\right\},$$
$$2\cos\left\{2(ac+bd)\frac{(n-1)\pi}{\hat{p}_n}+8\arctan\frac{c}{d}\right\},\ldots,$$

and

$$2\cos\left\{2(ac+bd)\frac{(n-\hat{p}_n+1)\pi}{\hat{p}_n}+8\arctan\frac{c}{d}\right\}$$

will be greater than 1, and at least one will be less than -1. We choose two such expressions and denote them by

$$2\cos\left\{2(ac+bd)\frac{(n-g)\pi}{\hat{p}_n}+8\arctan\frac{c}{d}\right\} \tag{12.2.21}$$

and

$$2\cos\left\{2(ac+bd)\frac{(n-h)\pi}{\hat{p}_n}+8\arctan\frac{c}{d}\right\}, \tag{12.2.22}$$

respectively.

The term arising from (12.2.21), by (12.2.17) and (12.2.18), is, as $n\to\infty$,

$$\frac{2}{C}\frac{2\cos\left\{2(ac+bd)\dfrac{(n-g)\pi}{\hat{p}_n}+8\arctan\dfrac{c}{d}\right\}e^{2(n-g)\pi/\hat{p}_n}}{\hat{p}_n^4}$$
$$\begin{aligned}
&> \frac{2}{C}\frac{e^{2(n-g)\pi/\hat{p}_n}}{\hat{p}_n^4}. \\
&= \frac{2}{C}\frac{e^{(2n\pi/\hat{p}_n)(1+o(1))}}{\hat{p}_n^4} \\
&= \frac{2}{C}\frac{e^{(4(\log n)/(1-\epsilon))(1+o(1))}}{\left\{\dfrac{\pi(1-\epsilon)}{2}\dfrac{n}{\log n}\right\}^4} \\
&= \frac{32}{C\pi^4(1-\epsilon)^4}\left(\log^4 n\right)\ n^{4\epsilon/(1-\epsilon)+o(1)},
\end{aligned}$$

which tends to ∞ as n tends to ∞. Similarly, the term that includes the expression in (12.2.22) tends to $-\infty$ as n tends to ∞. This concludes the proof of the theorem. □

Actually, we have proved a bit more, namely, we have proved that there are also terms in $\sum_n$ that are not in $\sum_n^{\ell(n)}$ and are arbitrarily close to $-\infty$.

12.3 An Upper Bound

We prove an upper bound for the number of terms in the series $\sum_n$ required to determine the value of the coefficient γ_n.

Let

$$u(n) := \left\{\frac{n(1+\epsilon)}{\log^{3/2} n\sqrt{\displaystyle\prod_{p\equiv 1\ (\mathrm{mod}\ 4)}\left(1-\frac{1}{p^2}\right)}}\right\},$$

where $0<\epsilon<1$ and p runs over the primes congruent to 1 modulo 4. Let $\sum_n^{u(n)}$ be the sum of the first $u(n)$ terms of $\sum_n$, where all of the summands corresponding to one value of μ together count as one "term."

Entry 12.3.1 (pp. 97–101). (*An Upper Bound for the Number of Terms Required*) *The coefficient* γ_n *defined in* (12.2.1) *is the nearest integer to* $\sum_n^{u(n)}$ *for* n *sufficiently large.*

Proof. A typical term of the series $\sum_n$ is of the form

$$\frac{2}{C}\frac{\{2\cos\{\cdots\}+\cdots+2\cos\{\cdots\}\}}{\mu^4}e^{2n\pi/\mu}, \tag{12.3.1}$$

where the number of expressions "$2\cos\{\cdots\}$" in the term is equal to the number of distinct representations of μ as a sum of two coprime squares. The number of (not necessarily distinct) representations of an integer m as a sum of two squares is (e.g., see [223, p. 167])

$$4\Bigg(\sum_{\substack{d|m\\ d\equiv 1\,(\mathrm{mod}\,4)}} 1 - \sum_{\substack{d|m\\ d\equiv 3\,(\mathrm{mod}\,4)}} 1\Bigg).$$

Thus the number of expressions "$2\cos\{\cdots\}$" in the term (12.3.1) is less than $4d(\mu)$, where $d(\mu)$ denotes the number of positive divisors of μ, and the term (12.3.1) itself is less than

$$\frac{16}{C}\frac{d(\mu)}{\mu^4}e^{2n\pi/\mu} \tag{12.3.2}$$

in modulus.

The value of μ corresponding to the final term in the series $\sum_n^{u(n)}$ is $\mu_{u(n)}$. We now estimate the sum of the terms of $\sum_n$ that are not in $\sum_n^{u(n)}$, that is, the terms that correspond to values of μ greater than $\mu_{u(n)}$.

By the definition of $c(m)$,

$$c(\mu_{u(m)}) = u(m).$$

Using a line of proof that is virtually identical to that for Entry 12.2.1, we find that

$$\mu_{u(m)} \sim \frac{2\pi(1+\epsilon)}{3}\frac{n}{\log n}. \tag{12.3.3}$$

By (12.3.2),

$$\begin{aligned}\Big|\sum\nolimits_n - \sum\nolimits_n^{u(n)}\Big| < \frac{16}{C}\Bigg\{ & \frac{d(\mu_{u(n)+1})}{\mu_{u(n)+1}^4}e^{2n\pi/\mu_{u(n)+1}} \\ & + \frac{d(\mu_{u(n)+2})}{\mu_{u(n)+2}^4}e^{2n\pi/\mu_{u(n)+2}} + \cdots\Bigg\}. \end{aligned} \tag{12.3.4}$$

By a result of Hardy and Ramanujan (e.g., see [223, p. 396]),

$$d(m) = m^{o(1)}$$

as m tends to ∞, so that, by (12.3.3), the right-hand side of (12.3.4) is equal to

$$\begin{aligned} & O\left(\frac{1}{\mu_{u(n)+1}^{4-o(1)}}e^{2n\pi/\mu_{u(n)+1}} + \frac{1}{\mu_{u(n)+2}^{4-o(1)}}e^{2n\pi/\mu_{u(n)+2}} + \cdots\right) \\ & \qquad = O\left(e^{2n\pi/\mu_{u(n)}}\left(\frac{1}{(\mu_{u(n)}+1)^{4-o(1)}} + \frac{1}{(\mu_{u(n)}+2)^{4-o(1)}} + \cdots\right)\right) \end{aligned}$$

$$\begin{aligned}
&= O\left(e^{2n\pi/\mu_{u(n)}} \frac{1}{\mu_{u(n)}^{3-o(1)}}\right) \\
&= O\left(e^{(3\log n/(1+\epsilon))(1-o(1))} \frac{1}{(n/\log n)^{3-o(1)}}\right) \\
&= O\left(n^{(3-o(1))/(1+\epsilon)} \frac{1}{(n/\log n)^{3-o(1)}}\right) \\
&= O\left(\frac{\log^3 n}{n^{3\epsilon/(1+\epsilon)-o(1)}}\right), \qquad (12.3.5)
\end{aligned}$$

by (12.3.3). Thus, from (12.3.4) and (12.3.3),

$$\left|\sum\nolimits_n - \sum\nolimits_n^{u(n)}\right| = o(1)$$

as n tends to ∞.

Since $\sum_n$ is an integer and the expression

$$\left|\sum\nolimits_n - \sum\nolimits_n^{u(n)}\right|$$

is less than one-half for n sufficiently large, we see that γ_n (which equals $\sum_n$) is the nearest integer to $\sum_n^{u(n)}$ for n sufficiently large, and thus we have proved the result. $\square$

In his letter to Hardy [244, pp. 97–101] that includes Entries 12.2.1 and 12.3.1, Ramanujan claims that the terms in (12.2.19) can be written as [244, p. 97]

$$\frac{2}{C}\frac{2\cos\left\{\dfrac{2n\pi}{\hat{p}_n} + 8\arctan\theta\right\} e^{2n\pi/\hat{p}_n}}{\hat{p}_n^4}, \qquad (12.3.6)$$

$$\frac{2}{C}\frac{2\cos\left\{\dfrac{2(n-1)\pi}{\hat{p}_n} + 8\arctan\theta\right\} e^{2(n-1)\pi/\hat{p}_n}}{\hat{p}_n^4}, \qquad (12.3.7)$$

etc. In other words, he assumes that there exist integers a, b, c, and d such that $ad - bc = 1$, $c^2 + d^2 = \hat{p}_n$, and $ac + bd \equiv \pm 1 \pmod{\hat{p}_n}$. This assumption does not seem to be correct.

For example, he asserts that the term with $\mu = 5$ can be written as

$$\frac{2}{C}\frac{2\cos\left\{\dfrac{2n\pi}{5} + 8\arctan 2\right\} e^{2n\pi/5}}{5^4}.$$

However, this is not true. By (11.5.7), we see that

$$W_5(n) = 2\cos\left\{(ac+bd)\frac{2n\pi}{5} + 8\arctan\frac{c}{d}\right\},$$

where a, b, c, $d \in \mathbb{Z}$, $ad - bc = 1$, and $c^2 + d^2 = 5$. Note that

$$(ad-bc)^2 + (ac+bd)^2 = (a^2+b^2)(c^2+d^2).$$

The right-hand side is divisible by 5, since $c^2 + d^2 = 5$. Since $ad - bc = 1$,

$$(ac+bd)^2 \equiv 4 \pmod{5},$$

so that

$$ac + bd \equiv 2, 3 \pmod{5}.$$

Thus, it is puzzling why Ramanujan writes

$$2\cos\left\{\frac{2n\pi}{5} + 8\arctan 2\right\}$$

in his expression for $W_5(n)$, because this would imply

$$ac + bd \equiv \pm 1 \pmod{5}.$$

13

Eisenstein Series and Modular Equations

13.1 Introduction

Recall that Ramanujan's three Eisenstein series $P(q)$, $Q(q)$, and $R(q)$ are defined by

$$P(q) := 1 - 24 \sum_{k=1}^{\infty} \frac{kq^k}{1-q^k}, \tag{13.1.1}$$

$$Q(q) := 1 + 240 \sum_{k=1}^{\infty} \frac{k^3 q^k}{1-q^k}, \tag{13.1.2}$$

and

$$R(q) := 1 - 504 \sum_{k=1}^{\infty} \frac{k^5 q^k}{1-q^k}, \tag{13.1.3}$$

where $|q| < 1$. On pages 44, 50, 51, and 53 in his lost notebook [244], Ramanujan offers 12 formulas for Eisenstein series. All are connected with modular equations of degree either 5 or 7.

In a wonderful paper [233] devoted to proving identities for Eisenstein series and incomplete elliptic integrals in Ramanujan's lost notebook, S. Raghavan and S.S. Rangachari employ the theory of modular forms in establishing proofs for all of Ramanujan's identities for Eisenstein series. Most of the identities give representations for certain Eisenstein series in terms of quotients of Dedekind eta functions, or, more precisely, Hauptmoduls. The very short proofs by Raghavan and Rangachari depend on the finite dimensions of the spaces of relevant modular forms, and therefore upon showing that a sufficient number of coefficients in the expansions about $q = 0$ of both sides of the proposed identities agree. Ramanujan evidently was unfamiliar with the theory of modular forms and most likely did not discover the identities by comparing coefficients.

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The purpose of this chapter is therefore to construct proofs in the spirit of Ramanujan's work. In fact, our proofs depend only on theorems found in Ramanujan's notebooks [243]. Admittedly, some of our algebraic manipulations are rather laborious, and we resorted at times to *Mathematica*. It is therefore clear to us that Ramanujan's calculations, at least in some cases, were more elegant than ours. We actually have devised two approaches. In Sections 13.3 and 13.4, we use the two methods, respectively, to prove Ramanujan's quintic identities. At the end of Section 13.3, we prove a first-order nonlinear "quintic" differential equation of Ramanujan satisfied by $P(q)$. In Section 13.5, we use the second approach, which is more constructive, to prove Ramanujan's septic identities. The new parameterizations for moduli of degree 7 in Section 13.5 appear to be more useful than those given in [54, Section 19]. In Section 13.6, we briefly describe two new first-order nonlinear "septic" differential equations for $P(q)$.

The content of this chapter is taken from a paper by Berndt, H.H. Chan, J. Sohn, and S.H. Son [67].

13.2 Preliminary Results

Define, after Ramanujan,

$$f(-q) := (q;q)_\infty =: e^{-2\pi iz/24}\eta(z), \qquad q = e^{2\pi iz}, \qquad \operatorname{Im} z > 0, \tag{13.2.1}$$

where η denotes the Dedekind eta function. We shall use the well-known transformation formula [54, p. 43, Entry 27(iii)]

$$\eta(-1/z) = \sqrt{z/i}\,\eta(z). \tag{13.2.2}$$

The Eisenstein series $Q(q)$ and $R(q)$ are modular forms of weights 4 and 6, respectively. In particular, they obey the easily proved and well-known transformation formulas [246, p. 136]

$$Q(e^{-2\pi i/z}) = z^4 Q(e^{2\pi iz}) \tag{13.2.3}$$

and

$$R(e^{-2\pi i/z}) = z^6 R(e^{2\pi iz}). \tag{13.2.4}$$

Our proofs below depend on modular equations. As usual, set

$$(a)_k := \frac{\Gamma(a+k)}{\Gamma(a)}$$

and

$$_2F_1(a,b;c;x) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} x^k, \qquad |x| < 1.$$

Suppose that, for some positive integer n,

$$\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\beta)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\beta)} = n\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}. \tag{13.2.5}$$

A modular equation of degree n is an equation involving α and β that is induced by (13.2.5). We often say that β *has degree n over α*. Also set

$$z_1 := {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\alpha) \qquad \text{and} \qquad z_n := {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;\beta). \tag{13.2.6}$$

The *multiplier* m is defined by

$$m := \frac{z_1}{z_n} = \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}, \tag{13.2.7}$$

where the last equality is a consequence of (13.2.6). When

$$q = \exp\left(-\pi\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;x)}\right) \tag{13.2.8}$$

and

$$z = {}_2F_1(\tfrac{1}{2},\tfrac{1}{2};1;x), \tag{13.2.9}$$

we have the "evaluations"

$$f(-q^2) = \sqrt{z}\, 2^{-1/3}\,(x(1-x)/q)^{1/12}, \tag{13.2.10}$$

$$Q(q^2) = z^4(1-x+x^2), \tag{13.2.11}$$

and

$$R(q^2) = z^6(1+x)(1-x/2)(1-2x). \tag{13.2.12}$$

These are, respectively, Entries 12(iii), 13(i), and 13(ii) in Chapter 17 of Ramanujan's second notebook [54, pp. 124, 126].

Next, we record some relations from the theory of modular equations of degree 5. Set

$$m = 1+2p, \qquad 0<p<2, \tag{13.2.13}$$

and

$$\rho = (m^3-2m^2+5m)^{1/2}. \tag{13.2.14}$$

Then [54, p. 284, equations (13.4), (13.5)]

$$\left(\frac{\alpha^5}{\beta}\right)^{1/8} = \frac{5\rho+m^2+5m}{4m^2}, \qquad \left(\frac{\beta^5}{\alpha}\right)^{1/8} = \frac{\rho-m-1}{4}, \tag{13.2.15}$$

$$\left(\frac{(1-\alpha)^5}{1-\beta}\right)^{1/8} = \frac{5\rho-m^2-5m}{4m^2}, \quad \text{and} \quad \left(\frac{(1-\beta)^5}{1-\alpha}\right)^{1/8} = \frac{\rho+m+1}{4}. \tag{13.2.16}$$

Furthermore [54, p. 288, Entry 14(ii)]

$$4\alpha(1-\alpha) = p\left(\frac{2-p}{1+2p}\right)^5 \tag{13.2.17}$$

and

$$4\beta(1-\beta) = p^5\left(\frac{2-p}{1+2p}\right). \tag{13.2.18}$$

Also, from Entry 14(iii) in Chapter 19 of Ramanujan's second notebook [54, p. 289],

$$1-2\beta = (1+p-p^2)\left(\frac{1+p^2}{1+2p}\right)^{1/2}. \tag{13.2.19}$$

We also need two modular equations of degree 5 from Entry 13(iv) of Chapter 18 in Ramanujan's second notebook [54, p. 281], namely,

$$m = 1 + 2^{4/3}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24} \tag{13.2.20}$$

and

$$\frac{5}{m} = 1 + 2^{4/3}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24}. \tag{13.2.21}$$

For Section 13.5, we need several modular equations of degree 7 found in Entries 19(i), (ii), (iii), and (vii) of Ramanujan's second notebook [54, pp. 314–315]. Thus, if β has degree 7 over α and m is the multiplier of degree 7,

$$(\alpha\beta)^{1/8} + \{(1-\alpha)(1-\beta)\}^{1/8} = 1, \tag{13.2.22}$$

$$m = -\frac{1 - 4\left(\dfrac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24}}{(\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}}, \tag{13.2.23}$$

$$\frac{7}{m} = \frac{1 - 4\left(\dfrac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/24}}{(\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}}, \tag{13.2.24}$$

$$\begin{aligned}\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8} &- \left(\frac{\beta^7}{\alpha}\right)^{1/8} \\ &= m\left(\left(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}\right)/2\right)^{1/2},\end{aligned} \tag{13.2.25}$$

$$\begin{aligned}\left(\frac{\alpha^7}{\beta}\right)^{1/8} &- \left(\frac{(1-\alpha)^7}{1-\beta}\right)^{1/8} \\ &= \frac{7}{m}\left(\left(1 + (\alpha\beta)^{1/2} + \{(1-\alpha)(1-\beta)\}^{1/2}\right)/2\right)^{1/2},\end{aligned} \tag{13.2.26}$$

and

$$m - \frac{7}{m} = 2\left((\alpha\beta)^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}\right) \\ \times \left(2 + (\alpha\beta)^{1/4} + \{(1-\alpha)(1-\beta)\}^{1/4}\right). \tag{13.2.27}$$

13.3 Quintic Identities: First Method

Entry 13.3.1 (p. 50). *For* $Q(q)$ *and* $f(-q)$ *defined by* (13.1.2) *and* (13.2.1), *respectively,*

$$Q(q) = \frac{f^{10}(-q)}{f^2(-q^5)} + 250qf^4(-q)f^4(-q^5) + 3125q^2\frac{f^{10}(-q^5)}{f^2(-q)} \tag{13.3.1}$$

and

$$Q(q^5) = \frac{f^{10}(-q)}{f^2(-q^5)} + 10qf^4(-q)f^4(-q^5) + 5q^2\frac{f^{10}(-q^5)}{f^2(-q)}. \tag{13.3.2}$$

Proof. It is slightly advantageous to first prove (13.3.2) with q replaced by q^2. To prove (13.3.2), we first write the right side of (13.3.2) as a function of p, where p is defined by (13.2.13).

By (13.2.10),

$$\begin{aligned} q^4\frac{f^{10}(-q^{10})}{f^2(-q^2)} &= q^4\frac{z_5^5 2^{-10/3}\left(\beta(1-\beta)/q^5\right)^{5/6}}{z_1 2^{-2/3}\left(\alpha(1-\alpha)/q\right)^{1/6}} \\ &= 2^{-8/3}\frac{z_5^4}{m}\left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/6}, \end{aligned} \tag{13.3.3}$$

where β has degree 5 over α, z_1 and z_5 are defined by (13.2.6), and m is the multiplier defined by (13.2.7). Using (13.2.15), (13.2.16), (13.2.14), and (13.2.13) in (13.3.3), we find that

$$\begin{aligned} q^4\frac{f^{10}(-q^{10})}{f^2(-q^2)} &= 2^{-8/3}\frac{z_5^4}{m}\left(\frac{\rho^2-(m+1)^2}{16}\right)^{4/3} \\ &= \frac{z_5^4(m-1)^4}{2^8 m} = \frac{z_5^4 p^4}{2^4(1+2p)}. \end{aligned} \tag{13.3.4}$$

Similarly, from (13.2.10), (13.2.7), (13.2.17), (13.2.18), and (13.2.13),

$$\frac{f^6(-q^2)}{q^2f^6(-q^{10})} = m^3\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2}$$

$$= m^3\left(\frac{2-p}{p(1+2p)}\right)^2 = \frac{(1+2p)(2-p)^2}{p^2}. \qquad (13.3.5)$$

Thus, from (13.3.4) and (13.3.5),

$$\begin{aligned}
&q^4\frac{f^{10}(-q^{10})}{f^2(-q^2)}\left(\frac{f^{12}(-q^2)}{q^4f^{12}(-q^{10})}+10\frac{f^6(-q^2)}{q^2f^6(-q^{10})}+5\right)\\
&=\frac{z_5^4p^4}{2^4(1+2p)}\left(\frac{(1+2p)^2(2-p)^4}{p^4}+10\frac{(1+2p)(2-p)^2}{p^2}+5\right)\\
&=\frac{z_5^4}{2^4(1+2p)}(16+32p-8p^5+4p^6)\\
&=z_5^4\left(1+\frac{p^5(-2+p)}{4(1+2p)}\right)\\
&=z_5^4(1-\beta(1-\beta))\\
&=Q(q^{10}),
\end{aligned}$$

where in the penultimate step we used (13.2.18), and in the last step we utilized (13.2.11). This completes the proof of (13.3.2).

To prove (13.3.1), we first rewrite (13.3.2) in terms of the Dedekind eta function, defined in (13.2.1). Accordingly,

$$Q(q^5) = \frac{\eta^{10}(5z)}{\eta^2(z)}\left(\left(\frac{\eta(z)}{\eta(5z)}\right)^{12}+10\left(\frac{\eta(z)}{\eta(5z)}\right)^6+5\right). \qquad (13.3.6)$$

We now transform (13.3.6) by means of (13.2.2) and (13.2.3) to deduce that

$$\begin{aligned}
(5z)^{-4}Q(e^{-2\pi i/(5z)}) &= \frac{\eta^{10}(-1/(5z))}{(5z/i)^5}\frac{(z/i)}{\eta^2(-1/z)}\\
&\times\left(\left(\frac{\eta(-1/z)}{\sqrt{z/i}}\frac{\sqrt{5z/i}}{\eta(-1/(5z))}\right)^{12}+10\left(\frac{\eta(-1/z)}{\sqrt{z/i}}\frac{\sqrt{5z/i}}{\eta(-1/(5z))}\right)^6+5\right),
\end{aligned}$$

or

$$\begin{aligned}
&Q(e^{-2\pi i/(5z)})\\
&\quad=\frac{\eta^{10}(-1/(5z))}{\eta^2(-1/z)}\left(5^5\left(\frac{\eta(-1/z)}{\eta(-1/(5z))}\right)^{12}+250\left(\frac{\eta(-1/z)}{\eta(-1/(5z))}\right)^6+1\right)\\
&\quad=5^5\frac{\eta^{10}(-1/z)}{\eta^2(-1/(5z))}+250\eta^4(-1/(5z))\eta^4(-1/z)+\frac{\eta^{10}(-1/(5z))}{\eta^2(-1/z)}.
\end{aligned}$$

If we set $q = e^{-2\pi i/(5z)}$ and use (13.2.1), the last equality takes the shape (13.3.1), and so this completes the proof of (13.3.1). $\square$

Entry 13.3.2 (p. 51). *For $f(-q)$ and $R(q)$ defined by* (13.2.1) *and* (13.1.3), *respectively,*

$$R(q) = \left(\frac{f^{15}(-q)}{f^3(-q^5)} - 500qf^9(-q)f^3(-q^5) - 15625q^2f^3(-q)f^9(-q^5)\right) \times\sqrt{1+22q\frac{f^6(-q^5)}{f^6(-q)}+125q^2\frac{f^{12}(-q^5)}{f^{12}(-q)}} \tag{13.3.7}$$

and

$$R(q^5) = \left(\frac{f^{15}(-q)}{f^3(-q^5)} + 4qf^9(-q)f^3(-q^5) - q^2f^3(-q)f^9(-q^5)\right) \times\sqrt{1+22q\frac{f^6(-q^5)}{f^6(-q)}+125q^2\frac{f^{12}(-q^5)}{f^{12}(-q)}}. \tag{13.3.8}$$

Proof. Our procedure is similar to that of the previous entry. We establish (13.3.8) first, but with q replaced by q^2.

By (13.2.10), (13.2.7), (13.2.17), and (13.2.18),

$$\frac{f^{15}(-q^2)}{f^3(-q^{10})} = \frac{z_1^{15/2}}{16z_5^{3/2}}\left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/4} = \frac{z_1^6m^{3/2}}{64}\left(\frac{2-p}{1+2p}\right)^6. \tag{13.3.9}$$

Hence, from (13.3.9), (13.3.5), (13.2.7), and (13.2.13),

$$\begin{aligned} F(q) &:= \frac{f^{15}(-q^2)}{f^3(-q^{10})}\left(1+4q^2\frac{f^6(-q^{10})}{f^6(-q^2)} - q^4\frac{f^{12}(-q^{10})}{f^{12}(-q^2)}\right) \\ &\quad\times\sqrt{1+22q^2\frac{f^6(-q^{10})}{f^6(-q^2)}+125q^4\frac{f^{12}(-q^{10})}{f^{12}(-q^2)}} \\ &= \frac{z_1^6m^{3/2}}{64}\left(\frac{2-p}{1+2p}\right)^6 \\ &\quad\times\left(1+4\frac{p^2}{(1+2p)(2-p)^2} - \frac{p^4}{(1+2p)^2(2-p)^4}\right) \\ &\quad\times\sqrt{1+22\frac{p^2}{(1+2p)(2-p)^2}+125\frac{p^4}{(1+2p)^2(2-p)^4}} \\ &= \frac{z_5^6}{8m^{3/2}}\left(4+8p-6p^2-6p^3+9p^4-5p^5+p^6\right) \\ &\quad\times\sqrt{4+8p+12p^2+12p^3+9p^4+4p^5+p^6} \\ &= \frac{z_5^6}{8m^{3/2}}(1+p-p^2)(4+4p-6p^2+4p^3-p^4) \\ &\quad\times\sqrt{(1+p^2)(4+8p+8p^2+4p^3+p^4)}. \end{aligned} \tag{13.3.10}$$

Using (13.2.19) and (13.2.13), we can write (13.3.10) in the form

$$\begin{aligned}
F(q) &= z_5^6(1-2\beta)\frac{4+4p-6p^2+4p^3-p^4}{8(1+2p)}\sqrt{4+8p+8p^2+4p^3+p^4}\\
&= z_5^6(1-2\beta)\frac{(4+4p-6p^2+4p^3-p^4)(2+2p+p^2)}{8(1+2p)}\\
&= z_5^6(1-2\beta)\frac{8+16p+2p^5-p^6}{8(1+2p)}\\
&= z_5^6(1-2\beta)\left(1+\frac{p^5(2-p)}{8(1+2p)}\right)\\
&= z_5^6(1-2\beta)\left(1+\tfrac{1}{2}\beta(1-\beta)\right)\\
&= z_5^6(1-2\beta)(1-\tfrac{1}{2}\beta)(1+\beta)\\
&= R(q^{10}),
\end{aligned} \tag{13.3.11}$$

where in the antepenultimate line we used (13.2.18), and in the last line we used (13.2.12). Combining (13.3.10) and (13.3.11), we deduce (13.3.8), but with q replaced by q^2.

The proof of (13.3.7) is almost exactly like the proof of (13.3.1), but of course, we use (13.2.4) instead of (13.2.3). □

The next two results are algebraic combinations of the pairs of representations in Entries 13.3.1 and 13.3.2.

Entry 13.3.3 (p. 51). *Let $A = Q(q)$ and $B = Q(q^5)$. Then*

$$\begin{aligned}
&\sqrt{A^2+94AB+625B^2}\\
&\quad = 12\sqrt{5}\left(\frac{f^{10}(-q)}{f^2(-q^5)}+26qf^4(-q)f^4(-q^5)+125q^2\frac{f^{10}(-q^5)}{f^2(-q)}\right).
\end{aligned} \tag{13.3.12}$$

Proof. Set

$$C=\frac{f^5(-q)}{f(-q^5)}, \quad D=qf^4(-q)f^4(-q^5), \quad \text{and} \quad E=q\frac{f^5(-q^5)}{f(-q)}. \tag{13.3.13}$$

Note that

$$CE = D. \tag{13.3.14}$$

Equalities (13.3.1) and (13.3.2) now take the shapes

$$A = C^2+250D+3125E^2 \quad \text{and} \quad B = C^2+10D+5E^2, \tag{13.3.15}$$

respectively, and the proposed equality (13.3.12) has the form

$$\sqrt{A^2+94AB+625B^2} = 12\sqrt{5}\left(C^2+26D+125E^2\right). \tag{13.3.16}$$

Substitute (13.3.15) into (13.3.16), square both sides, use (13.3.14), and with just elementary algebra, (13.3.16) is then verified. □

Entry 13.3.4 (p. 51). *Let $A = R(q)$ and $B = R(q^5)$. Then*

$$\begin{aligned}\sqrt{5(A+125B)^2-(126)^2AB}\\ &= 252\left(\frac{f^{10}(-q)}{f^2(-q^5)}+62qf^4(-q)f^4(-q^5)+125q^2\frac{f^{10}(-q^5)}{f^2(-q)}\right)\\ &\quad\times\sqrt{\frac{f^{10}(-q)}{f^2(-q^5)}+22qf^4(-q)f^4(-q^5)+125q^2\frac{f^{10}(-q^5)}{f^2(-q)}}.\end{aligned}\tag{13.3.17}$$

Proof. We employ the notation (13.3.13). Equalities (13.3.7) and (13.3.8) then may be written as, respectively,

$$A = \left(C^3-500CD-5^6DE\right)\sqrt{1+22E^2/D+125E^4/D^2}\tag{13.3.18}$$

and

$$B = \left(C^3+4CD-DE\right)\sqrt{1+22E^2/D+125E^4/D^2},\tag{13.3.19}$$

and the proposed equality (13.3.17) has the form

$$\begin{aligned}\sqrt{5(A+125B)^2-(126)^2AB} &= 252(C^2+62D+125E^2)\\ &\quad\times\sqrt{C^2+22D+125E^2}.\end{aligned}\tag{13.3.20}$$

Square (13.3.20), use (13.3.18), (13.3.19), and (13.3.14), and simplify to verify the truth of (13.3.20). □

Our next goal is to establish a differential equation satisfied by $P(q)$, defined by (13.1.1). We need two lemmas.

Lemma 13.3.1. *Recall that $Q(q)$ and $R(q)$ are defined by* (13.1.2) *and* (13.1.3), *respectively. Let*

$$u := q^{1/4}f(-q)f(-q^5) \qquad \text{and} \qquad \lambda := q\left(\frac{f(-q^5)}{f(-q)}\right)^6.\tag{13.3.21}$$

Then

$$Q(q) = u^4\left(\frac{1}{\lambda}+250+5^5\lambda\right)\tag{13.3.22}$$

and

$$R(q) = u^6\left(\frac{1}{\lambda}-500-5^6\lambda\right)\sqrt{\frac{1}{\lambda}+22+125\lambda}.\tag{13.3.23}$$

Proof. Identities (13.3.22) and (13.3.23) are obtained from (13.3.1) and (13.3.7), respectively. For example, by (13.3.1) and (13.3.21),

$$\begin{aligned}Q(q) &= qf^4(-q)f^4(-q^5)\left(\frac{f^6(-q)}{qf^6(-q^5)}+250+3125q\frac{f^6(-q^5)}{f^6(-q)}\right)\\ &= u^4\left(\frac{1}{\lambda}+250+5^5\lambda\right).\end{aligned}$$

□

Lemma 13.3.2. *Recall that $f(-q)$ is defined by* (13.2.1). *Then*

$$1+6\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-30\sum_{k=1}^{\infty}\frac{kq^{5k}}{1-q^{5k}}$$
$$=\sqrt{\frac{f^{12}(-q)+22qf^6(-q)f^6(-q^5)+125q^2f^{12}(-q^5)}{f^2(-q)f^2(-q^5)}}.$$

Lemma 13.3.2 is part of Entry 4(i) in Chapter 21 of Ramanujan's second notebook, and a proof is given in [54, p. 463]. We give here a new short proof, based on Lemma 13.3.1.

Proof. Using Ramanujan's differential equations [240, equation 30], [242, p. 142]

$$q\frac{dQ}{dq}=\frac{PQ-R}{3} \qquad \text{and} \qquad q\frac{dR}{dq}=\frac{PR-Q^2}{2}, \tag{13.3.24}$$

we deduce that

$$Q^3-R^2=3qR\frac{dQ}{dq}-2qQ\frac{dR}{dq}. \tag{13.3.25}$$

From (13.3.22) and (13.3.23), we find that

$$Q^3-R^2=1728\frac{u^{12}}{\lambda^2}, \tag{13.3.26}$$

$$\frac{dQ}{dq}=4u^3\left(\frac{1}{\lambda}+250+5^5\lambda\right)\frac{du}{dq}+u^4\left(-\frac{1}{\lambda^2}+5^5\right)\frac{d\lambda}{dq}, \tag{13.3.27}$$

and

$$\begin{aligned}\frac{dR}{dq}&=6u^5\left(\frac{1}{\lambda}-500-5^6\lambda\right)\sqrt{\frac{1}{\lambda}+22+125\lambda}\,\frac{du}{dq}\\&\quad-\frac{3u^6(1-152\lambda+5250\lambda^2+250000\lambda^3+1953125\lambda^4)}{2\lambda^3\sqrt{\frac{1}{\lambda}+22+125\lambda}}\frac{d\lambda}{dq}.\end{aligned} \tag{13.3.28}$$

Using (13.3.22), (13.3.23), (13.3.27), and (13.3.28) to simplify the right-hand side of (13.3.25), we deduce that

$$Q^3-R^2=3qR\frac{dQ}{dq}-2qQ\frac{dR}{dq}=1728\frac{u^{10}}{\lambda^3\sqrt{\frac{1}{\lambda}+22+125\lambda}}q\frac{d\lambda}{dq}.$$

Combining this last equation with (13.3.26) yields

$$q\frac{d\lambda}{dq} = u^2\lambda\sqrt{\frac{1}{\lambda} + 22 + 125\lambda}. \tag{13.3.29}$$

On the other hand, by straightforward logarithmic differentiation,

$$q\frac{d\lambda}{dq} = \lambda\left(1 - 30\sum_{k=1}^{\infty}\frac{kq^{5k}}{1-q^{5k}} + 6\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}\right). \tag{13.3.30}$$

If we combine (13.3.29) and (13.3.30), we deduce Lemma 13.3.2. $\square$

Entry 13.3.5 (p. 44). *Let* $P(q)$ *be defined by* (13.1.1). *Then*

$$P(q) = \frac{f^5(-q)}{f(-q^5)}\left(\sqrt{1+22\lambda+125\lambda^2} - 30F(\lambda)\right) \tag{13.3.31}$$

and

$$P(q^5) = \frac{f^5(-q)}{f(-q^5)}\left(\sqrt{1+22\lambda+125\lambda^2} - 6F(\lambda)\right), \tag{13.3.32}$$

where λ *is defined in* (13.3.21), *and where* $F(\lambda)$ *satisfies the nonlinear first-order differential equation*

$$1 + \frac{25}{2}\lambda + \frac{5}{2\lambda}F^2(\lambda) = F'(\lambda)\sqrt{1+22\lambda+125\lambda^2}. \tag{13.3.33}$$

Proof. Assume that $F(\lambda)$ is defined by (13.3.31), so that (13.3.31) is trivially true. By (13.1.1) and Lemma 13.3.2, we have

$$\frac{5P(q^5) - P(q)}{4} = \frac{f^5(-q)}{f(-q^5)}\sqrt{1+22\lambda+125\lambda^2}, \tag{13.3.34}$$

with λ defined by (13.3.21). If we substitute (13.3.31) into (13.3.34) and solve for $P(q^5)$, we deduce (13.3.32). It remains to prove that $F(\lambda)$ satisfies the differential equation (13.3.33).

From (13.3.24), (13.3.22), (13.3.23), and (13.3.29), we find that, with the prime $'$ denoting differentiation with respect to q,

$$P(q) = 12q\frac{u'}{u} - 2\frac{u^2}{\sqrt{\lambda}}\sqrt{1+22\lambda+125\lambda^2}. \tag{13.3.35}$$

Differentiating (13.3.35) with the help of (13.3.29), we deduce that

$$q\frac{dP}{dq} = -u^4\frac{125\lambda^2-1}{\lambda} - 4\frac{u^2}{\sqrt{\lambda}}\left(q\frac{u'}{u}\right)\sqrt{1+22\lambda+125\lambda^2} + 12q\left(q\frac{u'}{u}\right)'. \tag{13.3.36}$$

Next, using another differential equation of Ramanujan [240, equation (30)], [242, p. 142],

$$q\frac{dP}{dq} = \frac{P^2 - Q}{12}, \tag{13.3.37}$$

(13.3.22), (13.3.35), and (13.3.36), we conclude that

$$12q\left(q\frac{u'}{u}\right)' - 12\left(q\frac{u'}{u}\right)^2 = -\frac{3}{4}\frac{u^4}{\lambda}\left(1+125\lambda^2+18\lambda\right). \tag{13.3.38}$$

We now identify Ramanujan's function $F(\lambda)$. Comparing (13.3.31) and (13.3.35), we conclude that

$$F(\lambda) = -\frac{2}{5}q\frac{u'}{u}\frac{\sqrt{\lambda}}{u^2} + \frac{1}{10}\sqrt{1+22\lambda+125\lambda^2}. \tag{13.3.39}$$

Rewriting (13.3.39) in the form

$$\frac{F(\lambda)}{\sqrt{\lambda}} - \frac{1}{10\sqrt{\lambda}}\sqrt{1+22\lambda+125\lambda^2} = -\frac{2}{5}q\frac{u'}{u}\frac{1}{u^2}, \tag{13.3.40}$$

and differentiating with respect to q, we deduce that

$$\begin{aligned}
-\frac{1}{2}\frac{u^2}{\sqrt{\lambda}}&\sqrt{\frac{1+22\lambda+125\lambda^2}{\lambda}}F(\lambda) + \frac{1}{\sqrt{\lambda}}q\frac{dF(\lambda)}{dq} - \frac{u^2}{20}\frac{125\lambda^2-1}{\lambda} \\
&= -\frac{2}{5}\left\{q\left(q\frac{u'}{u}\right)'\frac{1}{u^2} - \frac{2}{u^2}\left(q\frac{u'}{u}\right)^2\right\}.
\end{aligned} \tag{13.3.41}$$

Using (13.3.38) and (13.3.40), we may rewrite the right-hand side of (13.3.41) and deduce that

$$\begin{aligned}
-\frac{1}{2}\frac{u^2}{\sqrt{\lambda}}&\sqrt{\frac{1+22\lambda+125\lambda^2}{\lambda}}F(\lambda) + \frac{1}{\sqrt{\lambda}}q\frac{dF(\lambda)}{dq} - \frac{u^2}{20}\frac{125\lambda^2-1}{\lambda} \\
&= u^2\frac{1+18\lambda+125\lambda^2}{40\lambda} + \frac{5}{2}u^2\frac{F^2(\lambda)}{\lambda} + u^2\frac{1+22\lambda+125\lambda^2}{40\lambda} \\
&\quad - u^2\frac{1}{2\lambda}F(\lambda)\sqrt{1+22\lambda+125\lambda^2}.
\end{aligned} \tag{13.3.42}$$

Simplifying (13.3.42) with the use of (13.3.29), we deduce Ramanujan's differential equation (13.3.33). □

13.4 Quintic Identities: Second Method

The alternative method to proving Entries 13.3.1, 13.3.2, and 13.3.5 that we present in this section is more constructive than that in Section 13.3, but although no less elementary, is perhaps slightly more removed from procedures that Ramanujan might have employed. On the other hand, the method here is more amenable to proving further theorems of this sort, especially if one does not know their formulations beforehand.

We begin by introducing some simplifying notation and making some useful preliminary calculations. Set

$$p_1 := \left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/24}, \tag{13.4.1}$$

$$p_2 := \left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/24}, \tag{13.4.2}$$

and

$$C := \frac{\sqrt{z_5^5}}{16\sqrt{z_1}}. \tag{13.4.3}$$

Observe that, by (13.2.20) and (13.2.21), respectively,

$$p_1 = \frac{m-1}{2^{4/3}} \tag{13.4.4}$$

and

$$p_2 = \frac{5-m}{2^{4/3}m}. \tag{13.4.5}$$

It follows that

$$\alpha(1-\alpha) = p_1p_2^5 = \frac{m-1}{2^{4/3}}\left(\frac{5-m}{2^{4/3}m}\right)^5 = -\frac{(m-1)(m-5)^5}{16^2m^5} \tag{13.4.6}$$

and

$$\beta(1-\beta) = p_1^5p_2 = \left(\frac{m-1}{2^{4/3}}\right)^5\frac{5-m}{2^{4/3}m} = -\frac{(m-1)^5(m-5)}{16^2m}. \tag{13.4.7}$$

We also note that, by (13.4.3),

$$\frac{z_1^4}{16^2m^5} = \frac{z_1^4}{16^2(z_1/z_5)^5} = \frac{z_5^5}{16^2z_1} = C^2 \tag{13.4.8}$$

and

$$\frac{z_5^4}{16^2m} = \frac{z_5^4}{16^2(z_1/z_5)} = \frac{z_5^5}{16^2z_1} = C^2. \tag{13.4.9}$$

Since, by (13.4.3),

$$C^3 = \frac{\sqrt{z_5^{15}}}{16^3\sqrt{z_1^3}} = \frac{z_5^6}{16^3}\frac{1}{\sqrt{(z_1/z_3)^3}} = \frac{z_5^6}{16^3m\sqrt{m}},$$

we find that

$$\frac{z_1^6}{16^3m^6} = \frac{z_1^6}{16^3(z_1^6/z_5^6)} = \frac{z_5^6}{16^3} = C^3m\sqrt{m}. \tag{13.4.10}$$

We shall use (13.4.8)–(13.4.10) in our alternative proofs of Entries 13.3.1 and 13.3.2.

In view of (13.3.1), it is natural to introduce abbreviated notation for certain quotients of eta functions. Our goal is to represent these quotients as polynomials in the multiplier m. First, by (13.2.10), (13.4.3), (13.4.2), and (13.4.5),

$$\begin{aligned} r_1 :&= \frac{f^5(-q^2)}{f(-q^{10})} = \frac{\sqrt{z_1^5}\,2^{-5/3}(\alpha(1-\alpha)/q)^{5/12}}{\sqrt{z_5}\,2^{-1/3}(\beta(1-\beta)/q^5)^{1/12}} \\ &= \frac{\sqrt{z_5^5 z_1^3}}{2^{4/3}\sqrt{z_1 z_5^3}} \left(\frac{\alpha^5(1-\alpha)^5}{\beta(1-\beta)}\right)^{1/12} \\ &= \frac{16C}{2^{4/3}} m^3 p_2^2 = \frac{16C}{2^{4/3}} m^3 \left(\frac{5-m}{2^{4/3}m}\right)^2 \\ &= Cm(m-5)^2, \end{aligned} \tag{13.4.11}$$

and, by (13.2.10), (13.4.3), (13.4.1), and (13.4.4),

$$\begin{aligned} r_2 :&= q^2\frac{f^5(-q^{10})}{f(-q^2)} = q^2\frac{\sqrt{z_5^5}\,2^{-5/3}(\beta(1-\beta)/q^5)^{5/12}}{\sqrt{z_1}\,2^{-1/3}(\alpha(1-\alpha)/q)^{1/12}} \\ &= \frac{\sqrt{z_5^5}}{2^{4/3}\sqrt{z_1}} \left(\frac{\beta^5(1-\beta)^5}{\alpha(1-\alpha)}\right)^{1/12} \\ &= \frac{16C}{2^{4/3}} p_1^2 = \frac{16C}{2^{4/3}} \left(\frac{m-1}{2^{4/3}}\right)^2 \\ &= C(m-1)^2. \end{aligned} \tag{13.4.12}$$

Hence, by (13.4.11) and (13.4.12),

$$r_1 r_2 = q^2 f^4(-q^2) f^4(-q^{10}) = C^2 m(m-5)^2(m-1)^2. \tag{13.4.13}$$

The following lemma will be very useful.

Lemma 13.4.1. *Let*

$$g(m) := C^2\left(\sum_{k=0}^{6} c_k m^k\right).$$

If furthermore, we set, for some numbers $x_1, x_2,$ and x_3,

$$g(m) = x_1 r_1^2 + x_2 r_1 r_2 + x_3 r_2^2,$$

then

$$x_1 = c_6, \quad x_2 = c_5 + 20c_6, \quad \text{and} \quad x_3 = c_0.$$

Proof. Since, by (13.4.11)–(13.4.13),

$$\begin{aligned}
x_1r_1^2 &+ x_2r_1r_2 + x_3r_2^2 \\
&= C^2\big(x_3 + m(25x_2 - 4x_3) + m^2(625x_1 - 60x_2 + 6x_3) \\
&\quad + m^3(-500x_1 + 46x_2 - 4x_3) + m^4(150x_1 - 12x_2 + x_3) \\
&\quad + m^5(-20x_1 + x_2) + m^6x_1\big),
\end{aligned}$$

by matching the coefficients of m^k, $k = 0, \dots, 6$, we find that

$$\begin{aligned}
c_0 &= x_3, \\
c_1 &= 25x_2 - 4x_3, \\
c_2 &= 625x_1 - 60x_2 + 6x_3, \\
c_3 &= -500x_1 + 46x_2 - 4x_3, \\
c_4 &= 150x_1 - 12x_2 + x_3, \\
c_5 &= -20x_1 + x_2, \\
c_6 &= x_1.
\end{aligned}$$

Therefore, if the system above is not overdetermined, then $g(m)$ can be expressed as a linear combination of r_1^2, r_1r_2, and r_2^2. By solving the linear system of equations

$$\begin{aligned}
c_0 &= x_3, \\
c_5 &= -20x_1 + x_2, \\
c_6 &= x_1,
\end{aligned}$$

for x_1, x_2, and x_3, and noting that c_1, c_2, c_3, and c_4 are then uniquely determined, we complete the proof. □

We are now ready for our second proof of Entry 13.3.1.

Proof of Entry 13.3.1. By (13.2.11), (13.4.6), and (13.4.8),

$$\begin{aligned}
Q(q^2) &= z_1^4\big(1 - \alpha(1-\alpha)\big) \\
&= z_1^4\left(1 + \frac{(m-1)(m-5)^5}{16^2m^5}\right) \\
&= \frac{z_1^4}{16^2m^5}\big(16^2m^5 + (m-1)(m-5)^5\big) \\
&= C^2(m^6 + 230m^5 + \cdots + 5^5) \\
&= r_1^2 + 2\cdot 5^3r_1r_2 + 5^5r_2^2,
\end{aligned}$$

upon the use of Lemma 13.4.1. Replacing q^2 by q, we complete the proof of (13.3.1).

By (13.2.11), (13.4.7), and (13.4.9),

$$\begin{aligned}
Q(q^{10}) &= z_5^4\bigl(1-\beta(1-\beta)\bigr)\\
&= z_5^4\left(1+\frac{(m-1)^5(m-5)}{16^2m}\right)\\
&= \frac{z_5^4}{16^2m}\bigl(16^2m+(m-1)^5(m-5)\bigr)\\
&= C^2(m^6-10m^5+\cdots+5)\\
&= r_1^2+10r_1r_2+5r_2^2,
\end{aligned}$$

by an application of Lemma 13.4.1. Replacing q^2 by q, we complete the proof of (13.3.2). □

For the proof of Entry 13.3.2, it will be convenient to define

$$\begin{aligned}
D &:= m^2-2m+5,\\
E &:= m^2+2m+5,\\
F &:= m^2+20m-25,
\end{aligned}$$

and

$$G := m^2-4m-1.$$

Solving (13.4.6) and (13.4.7) and using the notation above, we deduce that

$$\alpha = \frac{1}{2}+\frac{\sqrt{D/m}F}{16m^2} \tag{13.4.14}$$

and

$$\beta = \frac{1}{2}+\frac{\sqrt{D/m}G}{16}. \tag{13.4.15}$$

(See also [54, p. 289, equation (14.2); p. 290, equation (14.4)].)

Using the notation above and Lemma 13.4.1, we may readily deduce the following lemma.

Lemma 13.4.2. *For D, E, F, and G defined above, for C defined by* (13.4.3), *and for r_1 and r_2, defined in* (13.4.11) *and* (13.4.12), *respectively, we have*

$$C^2DE^2 = r_1^2+22r_1r_2+5^3r_2^2,$$

$$C^2F(m^4-540m^3+1350m^2-14\cdot5^3m+5^4) = r_1^2-4\cdot5^3r_1r_2-5^6r_2^2,$$

and

$$C^2G(m^4-12m^3+54m^2-108m+1) = r_1^2+4r_1r_2-r_2^2.$$

Proof of Entry 13.3.2. By (13.2.12), (13.4.14), (13.4.10), and Lemma 13.4.2,

$$\begin{aligned}
R(q^2) &= z_1^6(1+\alpha)(1-\alpha/2)(1-2\alpha)\\
&= z_1^6\frac{\sqrt{D/m}F}{16^3m^6}(\sqrt{D/m}F-24m^2)(\sqrt{D/m}F+24m^2)\\
&= \frac{z_1^6}{16^3m^6}\sqrt{D/m}F\big((D/m)F^2-24^2m^4\big)\\
&= (C^3m\sqrt{m})\sqrt{D/m}F\left(E\left(m^4-540m^3+1350m^2\right.\right.\\
&\qquad\qquad \left.\left.-14\cdot 5^3m+5^4\right)/m\right)\\
&= \sqrt{C^2DE^2}\big(C^2F(m^4-540m^3+1350m^2-14\cdot 5^3m+5^4)\big)\\
&= \sqrt{r_1^2+22r_1r_2+5^3r_2^2}\cdot(r_1^2-4\cdot 5^3r_1r_2-5^6r_2^2)\\
&= \sqrt{(r_1^2+22r_1r_2+5^3r_2^2)/r_1^2}\cdot r_1(r_1^2-4\cdot 5^3r_1r_2-5^6r_2^2).
\end{aligned}$$

Replacing q^2 by q, we complete the proof of (13.3.7).

By (13.2.12), (13.4.15), (13.4.10), and Lemma 13.4.2,

$$\begin{aligned}
R(q^{10}) &= z_5^6(1+\beta)(1-\beta/2)(1-2\beta)\\
&= z_5^6\frac{\sqrt{D/m}G}{16^3}(\sqrt{D/m}G-24)(\sqrt{D/m}G+24)\\
&= \frac{z_5^6}{16^3}\sqrt{D/m}G\big((D/m)G^2-24^2\big)\\
&= (C^3m\sqrt{m})\sqrt{D/m}G\left(E\left(m^4-12m^3\right.\right.\\
&\qquad\qquad \left.\left.+54m^2-108m+1\right)/m\right)\\
&= \sqrt{C^2DE^2}\big(C^2G(m^4-12m^3+54m^2-108m+1)\big)\\
&= \sqrt{r_1^2+22r_1r_2+5^3r_2^2}\cdot(r_1^2+4r_1r_2-r_2^2)\\
&= \sqrt{(r_1^2+22r_1r_2+5^3r_2^2)/r_1^2}\cdot r_1(r_1^2+4r_1r_2-r_2^2).
\end{aligned}$$

Replacing q^2 by q, we complete the proof of (13.3.8). □

We now give an alternative proof of Entry 13.3.5. Recall that λ is defined in (13.3.21). For convenience, define

$$H := \sqrt{1+22\lambda+5^3\lambda^2} \tag{13.4.16}$$

and

$$J := \frac{f^5(-q)}{f(-q^5)}. \tag{13.4.17}$$

Then equation (13.3.1) can be written in the form

$$Q(q) = J^2(1 + 2 \cdot 5^3\lambda + 5^5\lambda^2), \tag{13.4.18}$$

and (13.3.34) takes the shape

$$5P(q^5) = P(q) + 4HJ. \tag{13.4.19}$$

Furthermore, (13.3.29) may be written as

$$\frac{d\lambda}{dq} = \frac{\lambda HJ}{q}. \tag{13.4.20}$$

By logarithmic differentiation, we deduce that

$$\begin{aligned}
\frac{1}{J}\frac{dJ}{dq} &= 5\sum_{k=1}^{\infty}\frac{(-k)q^{k-1}}{1-q^k} - \sum_{k=1}^{\infty}\frac{(-5k)q^{5k-1}}{1-q^{5k}} \\
&= \frac{1}{24q}\big(5P(q) - 5P(q^5)\big) \\
&= \frac{1}{24q}\Big(5P(q) - \big(P(q) + 4HJ\big)\Big) \\
&= \frac{1}{6q}\big(P(q) - HJ\big),
\end{aligned}$$

or

$$P(q) = JH + \frac{6q}{J}\frac{dJ}{dq} = J\left(H + \frac{6q}{J^2}\frac{dJ}{dq}\right). \tag{13.4.21}$$

Now define

$$\mathfrak{F} := -\frac{q}{5J^2}\frac{dJ}{dq}.$$

Then, by (13.4.21),

$$P(q) = J\,(H - 30\mathfrak{F}) \tag{13.4.22}$$

and

$$\frac{dJ}{dq} = -\frac{5J^2\mathfrak{F}}{q}. \tag{13.4.23}$$

Differentiating (13.4.16) with respect to λ, we find that

$$H'(\lambda) = \frac{22 + 2\cdot 5^3\lambda}{2\sqrt{1 + 22\lambda + 5^3\lambda^2}} = \frac{11 + 5^3\lambda}{H}.$$

Using this, (13.4.22), (13.4.23), and (13.4.20), we deduce that

$$\begin{aligned}
q\frac{dP}{dq} &= q\frac{d}{dq}\big(J(H - 30\mathfrak{F})\big) \\
&= q\frac{dJ}{dq}(H - 30\mathfrak{F}) + qJ\frac{d\lambda}{dq}\frac{d}{d\lambda}(H - 30\mathfrak{F}) \\
&= q\left(-\frac{5J^2\mathfrak{F}}{q}\right)(H - 30\mathfrak{F}) + qJ\left(\frac{\lambda HJ}{q}\right)(H'(\lambda) - 30\mathfrak{F}'(\lambda))
\end{aligned}$$

$$= J^2(-5\mathfrak{F}H + 150\mathfrak{F}^2 + 11\lambda + 5^3\lambda^2 - 30\lambda\mathfrak{F}'(\lambda)H). \tag{13.4.24}$$

On the other hand, by (13.4.22) and (13.4.18),

$$\begin{aligned}
&\frac{1}{12}\big(P^2(q) - Q(q)\big) \qquad\qquad (13.4.25)\\
&\quad = \frac{1}{12}\big(J^2(H-30\mathfrak{F})^2 - J^2(1+2\cdot 5^3\lambda + 5^5\lambda^2)\big)\\
&\quad = \frac{J^2}{12}\left(\left(\sqrt{1+22\lambda+5^3\lambda^2}\right)^2 - 60\mathfrak{F}H + 30^2\mathfrak{F}^2 - (1+250\lambda+5^5\lambda^2)\right)\\
&\quad = J^2(-5\mathfrak{F}H + 75\mathfrak{F}^2 - 19\lambda - 2\cdot 5^3\lambda^2).
\end{aligned}$$

Equating (13.4.24) and (13.4.25) by (13.3.37), we arrive at

$$\mathfrak{F}'(\lambda)H = 1 + \frac{25}{2}\lambda + \frac{5}{2\lambda}\mathfrak{F}^2,$$

which is (13.3.33).

By (13.4.19) and (13.4.22), we deduce that

$$P(q^5) = \frac{1}{5}\big(P(q) + 4HJ\big) = \frac{J}{5}(H - 30\mathfrak{F} + 4H) = J(H - 6\mathfrak{F}),$$

which completes the proof of (13.3.32).

13.5 Septic Identities

Entry 13.5.1 (p. 53). *For* $|q| < 1$,

$$\begin{aligned}
Q(q) = &\left(\frac{f^7(-q)}{f(-q^7)} + 5\cdot 7^2 q f^3(-q)f^3(-q^7) + 7^4q^2\frac{f^7(-q^7)}{f(-q)}\right) \\
&\times\left(\frac{f^7(-q)}{f(-q^7)} + 13qf^3(-q)f^3(-q^7) + 49q^2\frac{f^7(-q^7)}{f(-q)}\right)^{1/3}
\end{aligned}\tag{13.5.1}$$

and

$$\begin{aligned}
Q(q^7) = &\left(\frac{f^7(-q)}{f(-q^7)} + 5qf^3(-q)f^3(-q^7) + q^2\frac{f^7(-q^7)}{f(-q)}\right) \\
&\times\left(\frac{f^7(-q)}{f(-q^7)} + 13qf^3(-q)f^3(-q^7) + 49q^2\frac{f^7(-q^7)}{f(-q)}\right)^{1/3}.
\end{aligned}\tag{13.5.2}$$

We shall prove these identities with q replaced by q^2.

For convenience, define

$$C := \frac{\sqrt{z_1z_7}}{4}, \tag{13.5.3}$$

$$p_1 := 4\left(\frac{\beta^7(1-\beta)^7}{\alpha(1-\alpha)}\right)^{1/24}, \tag{13.5.4}$$

and

$$p_2 := 4\left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/24}. \tag{13.5.5}$$

By (13.2.10), (13.2.7), and the definitions above,

$$\begin{aligned} r_1 := \frac{f^7(-q^2)}{f(-q^{14})} &= \frac{\sqrt{z_1^7}2^{-7/3}\big(\alpha(1-\alpha)/q\big)^{7/12}}{\sqrt{z_7}2^{-1/3}\big(\beta(1-\beta)/q^7\big)^{1/12}} \\ &= \frac{\sqrt{z_1^3 z_7^3}z_1^2}{4z_7^2}\left(\frac{\alpha^7(1-\alpha)^7}{\beta(1-\beta)}\right)^{1/12} = \frac{\sqrt{z_1^3 z_7^3}}{4}m^2\left(\frac{p_2}{4}\right)^2 \\ &= C^3 m^2 p_2^2. \end{aligned} \tag{13.5.6}$$

Furthermore,

$$p_1 p_2 = 16\big(\alpha\beta(1-\alpha)(1-\beta)\big)^{1/4}, \tag{13.5.7}$$

$$r_2 := q^2 f^3(-q^2) f^3(-q^{14}) = C^3 p_1 p_2, \tag{13.5.8}$$

and

$$r_3 := q^4 \frac{f^7(-q^{14})}{f(-q^2)} = C^3 \frac{p_1^2}{m^2}. \tag{13.5.9}$$

Thus,

$$r_1 + 5\cdot 7^2 r_2 + 7^4 r_3 = C^3\left(m^2 p_2^2 + 5\cdot 7^2 p_1 p_2 + \frac{7^4 p_1^2}{m^2}\right) \tag{13.5.10}$$

and

$$r_1 + 13 r_2 + 49 r_3 = C^3\left(m^2 p_2^2 + 13 p_1 p_2 + \frac{49 p_1^2}{m^2}\right). \tag{13.5.11}$$

Now define

$$T := (\alpha\beta)^{1/8} - (1-\alpha)^{1/8}(1-\beta)^{1/8}. \tag{13.5.12}$$

Then, by (13.2.23) and (13.2.24),

$$-T = \frac{1-p_1}{m} \tag{13.5.13}$$

and

$$7T = (1-p_2)m. \tag{13.5.14}$$

Eliminating T in (13.5.13) and (13.5.14), we deduce that

$$mp_2 + \frac{7p_1}{m} = m + \frac{7}{m}. \tag{13.5.15}$$

By (13.2.22) and (13.5.12),

$$(\alpha\beta)^{1/8} = \frac{1+T}{2} \tag{13.5.16}$$

and

$$(1-\alpha)^{1/8}(1-\beta)^{1/8} = \frac{1-T}{2}. \tag{13.5.17}$$

Thus, by (13.5.7),

$$p_1p_2 = 16\big((\alpha\beta)^{1/8}(1-\alpha)^{1/8}(1-\beta)^{1/8}\big)^2 = (1-T^2)^2. \tag{13.5.18}$$

By (13.2.27), (13.5.12), (13.5.16), and (13.5.17), we deduce that

$$m - \frac{7}{m} = 2T\left(2 + \left(\frac{1+T}{2}\right)^2 + \left(\frac{1-T}{2}\right)^2\right).$$

Rewriting this, we have the following lemma.

Lemma 13.5.1. *For the multiplier m, and T defined in* (13.5.12),

$$m - \frac{7}{m} = 5T + T^3. \tag{13.5.19}$$

Applying Lemma 13.5.1 repeatedly, one can derive the following expressions.

Lemma 13.5.2. *If m denotes the multiplier and T is defined in* (13.5.12), *then*

$$\begin{aligned}
m^2 &= 7 + m(5T + T^3),\\
m^3 &= (35T + 7T^3) + m(7 + 25T^2 + 10T^4 + T^6),\\
m^4 &= (49 + 175T^2 + 70T^4 + 7T^6)\\
&\quad + m(70T + 139T^3 + 75T^5 + 15T^7 + T^9),\\
\frac{1}{m} &= -\frac{1}{7}(T^3 + 5T) + \frac{1}{7}m,\\
\frac{1}{m^2} &= \frac{1}{49}(7 + 25T^2 + 10T^4 + T^6) + \frac{1}{49}m(-5T - T^3).
\end{aligned}$$

Lemma 13.5.3. *Let m and T be as in the previous two lemmas, and let p_1 and p_2 be defined by* (13.5.4) *and* (13.5.5), *respectively. Then*

$$m^2p_2^2 + 13p_1p_2 + \frac{49p_1^2}{m^2} = (3 + T^2)^3. \tag{13.5.20}$$

Proof. By (13.5.15), (13.5.19), and (13.5.18),

$$\begin{aligned}
m^2p_2^2 + 13p_1p_2 + \frac{49p_1^2}{m^2} &= \left(mp_2 + \frac{7p_1}{m}\right)^2 - p_1p_2 \\
&= \left(m + \frac{7}{m}\right)^2 - p_1p_2 \\
&= \left(m - \frac{7}{m}\right)^2 + 28 - p_1p_2 \\
&= (T^3 + 5T)^2 + 28 - (1 - T^2)^2 \\
&= (3 + T^2)^3,
\end{aligned}$$

which completes the proof. □

By Lemma 13.5.3 and (13.5.11), we find that

$$(r_1+13r_2+49r_3)^{1/3} = C\left(m^2p_2^2 + 13p_1p_2 + \frac{49p_1^2}{m^2}\right)^{1/3} = C(3+T^2). \quad (13.5.21)$$

By (13.5.14),

$$mp_2 = m - 7T. \quad (13.5.22)$$

By Lemma 13.5.3, (13.5.22), and (13.5.18),

$$\frac{49p_1^2}{m^2} = (3 + T^2)^3 - (m - 7T)^2 - 13(1 - T^2)^2. \quad (13.5.23)$$

With the help of Lemma 13.5.2, we can now express each of r_1, r_2, and r_3 in the form

$$f_1(T) + mf_2(T).$$

Lemma 13.5.4. *Let r_1, r_2, and r_3 be defined by* (13.5.6), (13.5.8), *and* (13.5.9), *respectively, and let m and T be as in the three previous lemmas. Then*

$$\begin{aligned}
r_1 &= C^3\left((49T^2 + 7) + m(T^3 - 9T)\right), \\
r_2 &= C^3\left((T^2 - 1)^2\right), \\
r_3 &= \frac{C^3}{49}\left((7 + 4T^2 - 4T^4 + T^6) + m(9T - T^3)\right).
\end{aligned}$$

Proof. Use (13.5.6) and (13.5.22) to deduce the formula for r_1; use (13.5.8) and (13.5.18) to prove the formula for r_2; and use (13.5.9) and (13.5.23) for the formula for r_3. □

Lemma 13.5.5. *For the multiplier m, and T defined in* (13.5.12),

$$\alpha = \frac{1}{16m}(1 + T)(21 + 8m - 21T + 7T^2 - 7T^3) \quad (13.5.24)$$

and

$$\beta = \frac{1}{16}(1+T)(8-3m+3mT-mT^2+mT^3). \tag{13.5.25}$$

Proof. By (13.5.16) and (13.5.17), we deduce that

$$\begin{aligned}\left(\frac{(1-\beta)^7}{1-\alpha}\right)^{1/8} &= \frac{1-\beta}{(1-\alpha)^{1/8}(1-\beta)^{1/8}} = \frac{1-\beta}{(1-T)/2}, \\ \left(\frac{\beta^7}{\alpha}\right)^{1/8} &= \frac{\beta}{(\alpha\beta)^{1/8}} = \frac{\beta}{(1+T)/2}, \\ \left(\frac{\alpha^7}{\beta}\right)^{1/8} &= \frac{\alpha}{(\alpha\beta)^{1/8}} = \frac{\alpha}{(1+T)/2},\end{aligned}$$

and

$$\left(\frac{(1-\alpha)^7}{1-\beta}\right)^{1/8} = \frac{1-\alpha}{(1-\alpha)^{1/8}(1-\beta)^{1/8}} = \frac{1-\alpha}{(1-T)/2}.$$

Using these identities, (13.5.16), and (13.5.17) in (13.2.26) and (13.2.25), and then solving the linear equations for α and β, we obtain (13.5.24) and (13.5.25). □

Lemma 13.5.6. *Let*

$$g(T) := C^3\left(\sum_{k=0}^{3} c_{2k}T^{2k} + m\sum_{k=0}^{1} d_{2k+1}T^{2k+1}\right).$$

If

$$g(T) = x_1r_1 + x_2r_2 + x_3r_3,$$

for some complex numbers x_1, x_2, and x_3, then

$$x_1 = d_3 + c_6, \quad x_2 = c_4 + 4c_6, \quad \text{and} \quad x_3 = 49c_6.$$

Proof. Since

$$\begin{aligned}&(x_1r_1 + x_2r_2 + x_3r_3)/C^3 \\ &\quad = \left(7x_1 + x_2 + \frac{1}{7}x_3\right) + \left(49x_1 - 2x_2 + \frac{4}{49}x_3\right)T^2 \\ &\qquad + \left(x_2 - \frac{4}{49}x_3\right)T^4 + \frac{1}{49}x_3T^6 \\ &\qquad + m\left\{\left(-9x_1 + \frac{9}{49}x_3\right)T + \left(x_1 - \frac{1}{49}x_3\right)T^3\right\},\end{aligned}$$

by Lemma 13.5.4, we deduce the equalities

$$\begin{aligned} c_4 &= x_2 - \frac{4}{49}x_3, \\ c_6 &= \frac{1}{49}x_3, \\ d_3 &= x_1 - \frac{1}{49}x_3. \end{aligned}$$

Thus, solving the linear system above for x_1, x_2, and x_3, we complete the proof. □

We are now ready to prove Entry 13.5.1.

Proof of (13.5.1). By (13.2.11), (13.5.3), (13.5.24), and Lemma 13.5.2,

$$\begin{aligned} Q(q^2) &= z_1^4(1-\alpha+\alpha^2) \\ &= \left(\frac{\sqrt{z_1z_7}}{4}\right)^4 (4^4m^2)(1-\alpha+\alpha^2) \\ &= C^4(3+T^2)(147+64m^2+112mT-245T^2-112mT^3 \\ &\qquad\qquad + 49T^4+49T^6) \\ &= C(3+T^2)\cdot C^3\Big(147+64\big(7+m(5T+T^3)\big)+112mT \\ &\qquad\qquad\qquad -245T^2-112mT^3+49T^4+49T^6\Big) \\ &= C(3+T^2)\cdot C^3\big(595-245T^2+49T^4+49T^6 \\ &\qquad\qquad\qquad + m(432T-48T^3)\big). \end{aligned}$$

Thus, applying Lemma 13.5.6, we find that $x_1 = 1, x_2 = 5\cdot 7^2$, and $x_3 = 7^4$.

Since, by Lemma 13.5.4,

$$r_1 + 5\cdot 7^2 r_2 + 7^4 r_3 = C^3\big(595-245T^2+49T^4+49T^6+m(432T-48T^3)\big),$$

we deduce that

$$\begin{aligned} Q(q^2) &= C(3+T^2)\cdot(r_1+5\cdot 7^2 r_2+7^4 r_3) \\ &= (r_1+13r_2+49r_3)^{1/3}(r_1+5\cdot 7^2 r_2+7^4 r_3), \end{aligned}$$

by (13.5.21). Thus, replacing q^2 by q, we complete the proof of (13.5.1). □

Proof of (13.5.2). By (13.2.11), (13.5.3), (13.5.25), and Lemma 13.5.2,

$$\begin{aligned} Q(q^{14}) &= z_7^4(1-\beta+\beta^2) \\ &= \left(\frac{\sqrt{z_1z_7}}{4}\right)^4 \frac{4^4}{m^2}(1-\beta+\beta^2) \\ &= C^4(3+T^2)\Big(\frac{64}{m^2}+\frac{16}{m}(-T+T^3)+3-5T^2+T^4+T^6\Big) \\ &= C^4(3+T^2)\Big(64\big(\frac{1}{49}(7+25T^2+10T^4+T^6) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{49}m(-5T-T^3)\big)+16\big(-\frac{1}{7}(T^3+5T)+\frac{1}{7}m\big)(-T+T^3)\\
&+3-5T^2+T^4+T^6\Big)\\
=C(3+T^2)\cdot\frac{C^3}{49}\big(&595+1915T^2+241T^4+T^6\\
&\qquad+m(-432T+48T^3)\big).
\end{aligned}
$$

Thus, by Lemma 13.5.6, $x_1=1, x_2=5$, and $x_3=1$.

Since, by Lemma 13.5.4,

$$
r_1+5r_2+r_3=\frac{C^3}{49}\big(595+1915T^2+241T^4+T^6+m(-432T+48T^3)\big),
$$

we deduce that

$$
\begin{aligned}
Q(q^{14})&=C(3+T^2)\cdot(r_1+5r_2+r_3)\\
&=(r_1+13r_2+49r_3)^{1/3}(r_1+5r_2+r_3),
\end{aligned}
$$

by (13.5.21).

Thus, upon replacing q^2 by q, we complete the proof of (13.5.2). □

Entry 13.5.2 (p. 53). *For* $|q|<1$,

$$
\begin{aligned}
R(q)=&\left(\frac{f^7(-q)}{f(-q^7)}-7^2(5+2\sqrt{7})qf^3(-q)f^3(-q^7)-7^3(21+8\sqrt{7})q^2\frac{f^7(-q^7)}{f(-q)}\right)\\
&\times\left(\frac{f^7(-q)}{f(-q^7)}-7^2(5-2\sqrt{7})qf^3(-q)f^3(-q^7)-7^3(21-8\sqrt{7})q^2\frac{f^7(-q^7)}{f(-q)}\right)
\end{aligned}
\tag{13.5.26}
$$

and

$$
\begin{aligned}
R(q^7)=&\left(\frac{f^7(-q)}{f(-q^7)}+(7+2\sqrt{7})qf^3(-q)f^3(-q^7)+(21+8\sqrt{7})q^2\frac{f^7(-q^7)}{f(-q)}\right)\\
&\times\left(\frac{f^7(-q)}{f(-q^7)}+(7-2\sqrt{7})qf^3(-q)f^3(-q^7)+(21-8\sqrt{7})q^2\frac{f^7(-q^7)}{f(-q)}\right).
\end{aligned}
\tag{13.5.27}
$$

We shall prove the identities with q replaced by q^2.

By straightforward calculations, we deduce the following lemma.

Lemma 13.5.7. *If*

$$
\begin{aligned}
C^6&\left(\sum_{k=0}^{6}c_{2k}T^{2k}+m\sum_{k=0}^{4}d_{2k+1}T^{2k+1}\right)\\
&=(r_1+x_2r_2+x_3r_3)(r_1+y_2r_2+y_3r_3),
\end{aligned}
$$

for some real numbers x_2, x_3, y_2, *and* y_3, *then*

$$\begin{aligned}
c_8 &= x_3 + y_3 + x_2 y_2 - \frac{6}{7^2}(x_2 y_3 + x_3 y_2) + \frac{24}{7^4} x_3 y_3, \\
c_{10} &= \frac{1}{7^2}(x_2 y_3 + x_3 y_2) - \frac{8}{7^4} x_3 y_3, \\
c_{12} &= \frac{1}{7^4} x_3 y_3, \\
d_1 &= -126 - 9(x_2 + y_2) + \frac{9}{7^2}(x_2 y_3 + x_3 y_2) + \frac{18}{7^3} x_3 y_3.
\end{aligned}$$

Proof of (13.5.26). By (13.2.12), (13.5.3), (13.5.24), and Lemma 13.5.2,

$$\begin{aligned}
\frac{1}{C^6} R(q^2) &= \frac{z_1^6}{C^6}(1+\alpha)(1-\alpha/2)(1-2\alpha) \\
&= \left(\frac{\sqrt{z_1 z_7}}{4C}\right)^6 m^3 4^6 (1+\alpha)(1-\alpha/2)(1-2\alpha) \\
&= -75411 - 95130T^2 - 1841T^4 + 3780T^6 - 1029T^8 - 2058T^{10} \\
&\quad - 343T^{12} + m(-82152T - 77344T^3 - 16816T^5 - 160T^7 + 344T^9).
\end{aligned}$$

If we use Lemma 13.5.7 to find real solutions (x_2, x_3, y_2, y_3) satisfying $x_2 \leq y_2$, we find that

$$\begin{aligned}
x_2 &= -7^2(5 + 2\sqrt{7}), \quad x_3 = -7^3(21 + 8\sqrt{7}), \\
y_2 &= -7^2(5 - 2\sqrt{7}), \quad y_3 = -7^3(21 - 8\sqrt{7}).
\end{aligned}$$

By Lemmas 13.5.2 and 13.5.4,

$$\begin{aligned}
&\frac{1}{C^6}\left(r_1 - 7^2(5 + 2\sqrt{7}) r_2 - 7^3(21 + 8\sqrt{7}) r_3\right) \\
&\qquad \times \left(r_1 - 7^2(5 - 2\sqrt{7}) r_2 - 7^3(21 - 8\sqrt{7}) r_3\right) \\
&\quad = -75411 - 95130T^2 - 1841T^4 + 3780T^6 - 1029T^8 - 2058T^{10} \\
&\qquad - 343T^{12} + m(-82152T - 77344T^3 - 16816T^5 - 160T^7 + 344T^9) \\
&\quad = \frac{1}{C^6} R(q^2).
\end{aligned}$$

Thus we complete the proof of (13.5.26) after replacing q^2 by q. □

Proof of (13.5.27). By (13.2.12), (13.5.3), (13.5.25), and Lemma 13.5.2,

$$\begin{aligned}
\frac{1}{C^6} R(q^{14}) &= \frac{z_7^6}{C^6}(1+\beta)(1-\beta/2)(1-2\beta) \\
&= \left(\frac{\sqrt{z_1 z_7}}{4C}\right)^6 \frac{4^6}{m^3}(1+\beta)(1-\beta/2)(1-2\beta)
\end{aligned}$$

$$= 75411 + 505890T^2 + 470713T^4 + 157644T^6 + 18645T^8 + 498T^{10} \\ - T^{12} + m(-82152T - 77344T^3 - 16816T^5 - 160T^7 + 344T^9).$$

If we use Lemma 13.5.7 to find real solutions (x_2, x_3, y_2, y_3) satisfying $x_2 \geq y_2$, we find that

$$x_2 = 7 + 2\sqrt{7}, \quad x_3 = 21 + 8\sqrt{7},$$
$$y_2 = 7 - 2\sqrt{7}, \quad y_3 = 21 - 8\sqrt{7}.$$

By Lemmas 13.5.2 and 13.5.4,

$$\begin{aligned}
&\frac{1}{C^6}\big(r_1 + (7+2\sqrt{7})r_2 + (21+8\sqrt{7})r_3\big) \\
&\qquad \times \big(r_1 + (7-2\sqrt{7})r_2 + (21-8\sqrt{7})r_3\big) \\
&\quad = 75411 + 505890T^2 + 470713T^4 + 157644T^6 + 18645T^8 + 498T^{10} \\
&\qquad - T^{12} + m(-82152T - 77344T^3 - 16816T^5 - 160T^7 + 344T^9) \\
&\quad = \frac{1}{C^6} R(q^{14}).
\end{aligned}$$

Thus we complete the proof of (13.5.27) after replacing q^2 by q. ☐

As we indicated at the close of Section 13.1, the proofs presented here depend only on theorems recorded by Ramanujan in his notebooks [243] and lost notebook [244]. S. Cooper and P.C. Toh [142] have also found proofs of all of Ramanujan's results in this chapter, and they too have employed only ideas that Ramanujan would have known. Different proofs of Ramanujan's identities for Eisenstein series, depending on the theory of elliptic functions, have been constructed by Z.-G. Liu [206], [207], [209], [211].

13.6 Septic Differential Equations

Concluding this chapter, we offer two new septic differential equations for $P(q)$, defined in (13.1.1). Both involve variations of the same variable, but one is connected with the beautiful identities in Entry 13.5.1, while the other is connected with an emerging alternative septic theory of elliptic functions, initially begun in a paper by Chan and Y.L. Ong [112].

Theorem 13.6.1. *For* $|q| < 1$,

$$P(q) = \left(\frac{f^7(-q)}{f(-q^7)}\right)^{2/3} \left((1 + 13\lambda + 49\lambda^2)^{2/3} - 28F(\lambda)\right) \tag{13.6.1}$$

and

$$P(q^7) = \left(\frac{f^7(-q)}{f(-q^7)}\right)^{2/3} \left((1+13\lambda+49\lambda^2)^{2/3} - 4F(\lambda)\right), \tag{13.6.2}$$

where

$$\lambda = q\frac{f^4(-q^7)}{f^4(-q)},$$

and where $F(\lambda)$ *satisfies the nonlinear first-order differential equation*

$$1 + \frac{28}{3}\lambda + \frac{7F^2(\lambda)}{3\lambda\sqrt[3]{1+13\lambda+49\lambda^2}} = F'(\lambda)\sqrt[3]{1+13\lambda+49\lambda^2}. \tag{13.6.3}$$

Connections with the septic theory of elliptic functions are made manifest in the next theorem.

Theorem 13.6.2. *Recall that* $P(q)$ *is defined in* (13.1.1). *Let*

$$z = \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+2n^2}$$

and define x *by*

$$\frac{1-x}{x} = \frac{1}{7q}\left(\frac{f(-q)}{f(-q^7)}\right)^4. \tag{13.6.4}$$

Then

$$P(q) = z^2(1+12F_1(x)) \qquad \textit{and} \qquad P(q^7) = z^2(1+\tfrac{12}{7}F_1(x)),$$

where $F_1(x)$ *satisfies the differential equation*

$$\frac{dF_1(x)}{dx}x(1-x) + F_1^2(x) + \frac{2}{3}F_1(x)\frac{x^2+13x}{7-x+x^2} + \frac{7}{9}\frac{x(3+x)}{7-x+x^2} = 0. \tag{13.6.5}$$

The differential equation of Theorem 13.6.1 was discovered by Raghavan and Rangachari [233] and can be deduced from (13.6.5) by setting

$$F_1(x) = -\frac{7}{3}F(\lambda)\left(\sqrt[3]{1+13\lambda+49\lambda^2}\right)^{-2}, \tag{13.6.6}$$

where λ is given in Theorem 13.6.1. Proofs of Theorem 13.6.1, Theorem 13.6.2, and the assertion immediately above can be found in the paper by Berndt, Chan, Sohn, and Son [67].

14

Series Representable in Terms of Eisenstein Series

14.1 Introduction

In his famous paper [240], [242, pp. 136–162], Ramanujan shows, among a multitude of beautiful theorems and conjectures, that various classes of infinite series can be represented as polynomials in the Eisenstein series P, Q, and R. In his lost notebook [244, pp. 188, 369], Ramanujan claims that two further classes of infinite series also can be represented in terms of P, Q, and R. Our task in this chapter is to establish these claims.

On page 188 of his lost notebook, Ramanujan examines the series

$$T_{2k} := T_{2k}(q) := 1+\sum_{n=1}^{\infty}(-1)^n \left\{(6n-1)^{2k}q^{n(3n-1)/2} + (6n+1)^{2k}q^{n(3n+1)/2}\right\}, \tag{14.1.1}$$

where $|q| < 1$. Note that the exponents $n(3n\pm1)/2$ are the generalized pentagonal numbers. Ramanujan records formulas for $T_{2k}, k = 1, 2, \ldots, 6$, in terms of the Eisenstein series

$$P(q) := 1 - 24\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}, \tag{14.1.2}$$

$$Q(q) := 1 + 240\sum_{k=1}^{\infty}\frac{k^3q^k}{1-q^k}, \tag{14.1.3}$$

and

$$R(q) := 1 - 504\sum_{k=1}^{\infty}\frac{k^5q^k}{1-q^k}, \tag{14.1.4}$$

where $|q| < 1$. Ramanujan's formulations of these formulas are cryptic. The first is given by Ramanujan in the form

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$$\frac{1-5^2q-7^2q^2+\cdots}{1-q-q^2+\cdots}=P.$$

In succeeding formulas, only the first two terms of the numerator are given, and in two instances the denominator is replaced by a dash —. At the bottom of the page, he gives the first five terms of a general formula for T_{2k}. Details and proofs for all of Ramanujan's claims about $T_{2k}(q)$ are given in Section 14.2

On page 369 of his lost notebook [244], Ramanujan briefly considers two classes of infinite series. One of the classes is considered in more detail on page 188, as described above. Ramanujan briefly considered the second class in Entry 35(i) of Chapter 16 in his second notebook [243], [54, pp. 61–62], where a recurrence relation is given in terms of members of yet a third class of infinite series. The approach indicated by Ramanujan on page 369 of his lost notebook, however, is neater and more direct, with the aforementioned third class of series not arising. In this chapter we also prove the claims about this second class of series, namely, the series $U_n(q)$, which we now define. For each nonnegative integer n,

$$U_n(q):=\frac{1}{(q;q)_\infty^3}\sum_{j=1}^{\infty}(-1)^{j-1}(2j-1)^{n+1}q^{j(j-1)/2}=:\frac{F_n(q)}{(q;q)_\infty^3}. \tag{14.1.5}$$

In Section 14.3, we establish Ramanujan's claims about $U_n(q)$. Just as the identities for $T_{2k}(q)$ can be regarded as generalizations of Euler's pentagonal number theorem, the identities for $U_{2k}(q)$ can be considered as generalizations of Jacobi's identity [54, p. 39, Entry 24(ii)]

$$(q;q)_\infty^3=\tfrac{1}{2}\sum_{n=-\infty}^{\infty}(-1)^n(2n+1)q^{n(n+1)/2}. \tag{14.1.6}$$

14.2 The Series $T_{2k}(q)$

In this section, we prove each of the seven formulas on page 188 and also note an interesting corollary. Keys to our proofs are the pentagonal number theorem [54, p. 36, Entry 22(iii)]

$$(q;q)_\infty=1+\sum_{n=1}^{\infty}(-1)^n\left\{q^{n(3n-1)/2}+q^{n(3n+1)/2}\right\}, \tag{14.2.1}$$

where $|q|<1$, and Ramanujan's famous differential equations [240], [242, p. 142]

$$q\frac{dP}{dq}=\frac{P^2-Q}{12},\qquad q\frac{dQ}{dq}=\frac{PQ-R}{3},\qquad\text{and}\qquad q\frac{dR}{dq}=\frac{PR-Q^2}{2}. \tag{14.2.2}$$

We now state Ramanujan's six formulas for T_{2k} followed by a corollary and his general formula. Another proof of Entry 14.2.1 can be found in Z.-G. Liu's paper [208, pp. 9–12].

Entry 14.2.1 (p. 188). *If T_{2k} is defined by* (14.1.1) *and P, Q, and R are defined by* (14.1.2)–(14.1.4)*, then*

$$\text{(i)}\quad \frac{T_2(q)}{(q;q)_\infty} = P,$$

$$\text{(ii)}\quad \frac{T_4(q)}{(q;q)_\infty} = 3P^2 - 2Q,$$

$$\text{(iii)}\quad \frac{T_6(q)}{(q;q)_\infty} = 15P^3 - 30PQ + 16R,$$

$$\text{(iv)}\quad \frac{T_8(q)}{(q;q)_\infty} = 105P^4 - 420P^2Q + 448PR - 132Q^2,$$

$$\text{(v)}\quad \frac{T_{10}(q)}{(q;q)_\infty} = 945P^5 - 6300P^3Q + 10080P^2R - 5940PQ^2 + 1216QR,$$

$$\text{(vi)}\quad \frac{T_{12}(q)}{(q;q)_\infty} = 10395P^6 - 103950P^4Q + 221760P^3R - 196020P^2Q^2 + 80256PQR - 2712Q^3 - 9728R^2.$$

The first formula has an interesting arithmetic interpretation.

Corollary 14.2.1. *For $n \geq 1$, let $\sigma(n) = \sum_{d|n} d$, and define $\sigma(0) = -\frac{1}{24}$. Let n denote a nonnegative integer. Then*

$$-24 \sum_{\substack{j+k(3k\pm1)/2=n \\ j,k\geq 0}} (-1)^k \sigma(j) = \begin{cases} (-1)^r(6r-1)^2, & \text{if } n = r(3r-1)/2, \\ (-1)^r(6r+1)^2, & \text{if } n = r(3r+1)/2, \\ 0, & \text{otherwise.} \end{cases} \tag{14.2.3}$$

Since $\sigma(j)$ is multiplicative, we note that $\sigma(j)$ is even except when j is a square or twice a square. Thus, from Corollary 14.2.1, we see that, unless $n = r(3r \pm 1)/2$, the number of representations of n as a sum of a square or twice a square and a generalized pentagonal number $k(3k \pm 1)/2$ is even. For example, if $n = 20$, then $20 = 8 + 12 = 18 + 2$.

Corollary 14.2.1 is, in fact, called Euler's identity and is usually stated in a slightly different form. For a combinatorial proof of (14.2.3), see a paper by I. Pak [226, pp. 59–60], and for a bijective proof of an equivalent formulation, see S. Kim's paper [192].

Entry 14.2.2 (p. 188). *Define the polynomials $f_{2k}(P, Q, R), k \geq 1$, by*

$$f_{2k}(P, Q, R) := \frac{T_{2k}(q)}{(q;q)_\infty}. \tag{14.2.4}$$

Then, for $k \geq 1$,

$$
\begin{aligned}
f_{2k}(P,Q,R) = 1 \cdot 3 \cdots (2k-1) \Big\{ & P^k - \frac{k(k-1)}{3} P^{k-2}Q \\
& + \frac{8k(k-1)(k-2)}{45} P^{k-3}R - \frac{11k(k-1)(k-2)(k-3)}{210} P^{k-4}Q^2 \\
& + \frac{152k(k-1)(k-2)(k-3)(k-4)}{14175} P^{k-5}QR + \cdots \Big\}.
\end{aligned}
\tag{14.2.5}
$$

The statement of Entry 14.2.2 is admittedly incomplete. The missing terms represented by $+\cdots$ contain all products $P^aQ^bR^c$ such that $2a+4b+6c=2k$. It would be extremely difficult to find a general formula for $f_{2k}(P,Q,R)$ that would give explicit representations for each coefficient of $P^{2a}Q^{4b}R^{6c}$.

Important in our proofs are the simple identities

$$
(6n \pm 1)^2 = 24\frac{n(3n \pm 1)}{2} + 1. \tag{14.2.6}
$$

Proof of Entry 14.2.1. Observe that

$$
\begin{aligned}
P(q) &= 1 + 24q\frac{d}{dq}\sum_{n=1}^{\infty}\log(1-q^n) \\
&= 1 + 24q\frac{d}{dq}\log(q;q)_\infty \\
&= 1 + 24q\frac{\frac{d}{dq}(q;q)_\infty}{(q;q)_\infty}.
\end{aligned}
\tag{14.2.7}
$$

Thus, using (14.2.1) and (14.2.6), we find that

$$
\begin{aligned}
(q;q)_\infty P(q) &= (q;q)_\infty + 24q\frac{d}{dq}\left(1 + \sum_{n=1}^{\infty}(-1)^n\left\{q^{n(3n-1)/2} + q^{n(3n+1)/2}\right\}\right) \\
&= (q;q)_\infty + 24\sum_{n=1}^{\infty}(-1)^n\left\{\frac{n(3n-1)}{2}q^{n(3n-1)/2} + \frac{n(3n+1)}{2}q^{n(3n+1)/2}\right\} \\
&= (q;q)_\infty + \sum_{n=1}^{\infty}(-1)^n\Big\{\left((6n-1)^2-1\right)q^{n(3n-1)/2} \\
&\qquad + \left((6n+1)^2-1\right)q^{n(3n+1)/2}\Big\} \\
&= (q;q)_\infty + \sum_{n=1}^{\infty}(-1)^n\left\{(6n-1)^2q^{n(3n-1)/2} + (6n+1)^2q^{n(3n+1)/2}\right\} \\
&\quad - (q;q)_\infty + 1 \\
&= T_2(q).
\end{aligned}
\tag{14.2.8}
$$

This completes the proof of (i).

In the proofs of the remaining identities of Entry 14.2.1, in each case, we apply the operator $24q\frac{d}{dq}$ to the preceding identity. In each proof we also use the identities

$$24q\frac{d}{dq}T_{2k}(q) = T_{2k+2}(q) - T_{2k}(q), \tag{14.2.9}$$

which follows from differentiation and the use of (14.2.6), and

$$24q\frac{d}{dq}(q;q)_\infty = T_2(q) - (q;q)_\infty, \tag{14.2.10}$$

which arose in the proof of (14.2.8).

We now prove (ii). Applying the operator $24q\frac{d}{dq}$ to (14.2.8) and using (14.2.9) and (14.2.10), we deduce that

$$P(q)\left(T_2(q) - (q;q)_\infty\right) + (q;q)_\infty 24q\frac{d}{dq}P(q) = T_4(q) - T_2(q).$$

Employing (i) to simplify and using the first differential equation in (14.2.2), we arrive at

$$P^2(q)(q;q)_\infty + 2\left(P^2(q) - Q(q)\right)(q;q)_\infty = T_4(q),$$

or

$$T_4 = (3P^2 - 2Q)(q;q)_\infty, \tag{14.2.11}$$

as desired.

To prove (iii), we apply the operator $24q\frac{d}{dq}$ to (14.2.11) and use (14.2.9) and (14.2.10) to deduce that

$$\begin{aligned}
T_6 - T_4 &= 24\left(6Pq\frac{dP}{dq} - 2q\frac{dQ}{dq}\right)(q;q)_\infty + (3P^2 - 2Q)\left(T_2 - (q;q)_\infty\right) \\
&= \left(12P(P^2 - Q) - 16(PQ - R)\right)(q;q)_\infty + (3P^2 - 2Q)(P - 1)(q;q)_\infty,
\end{aligned}$$

where we used (14.2.2) and (i). If we now employ (14.2.11) and simplify, we conclude that

$$T_6 = \left(15P^3 - 30PQ + 16R\right)(q;q)_\infty.$$

In general, by applying the operator $24q\frac{d}{dq}$ to T_{2k} and using (14.2.9) and (14.2.10), we find that

$$T_{2k+2} - T_{2k} = 24q\frac{d}{dq}f_{2k}(P,Q,R)\,(q;q)_\infty + f_{2k}(P,Q,R)(P - 1)(q;q)_\infty,$$

where we have used the notation (14.2.4). Then proceeding by induction while using the formula (14.2.4) for T_{2k}, we find that

$$\frac{T_{2k+2}}{(q;q)_\infty} = 24q\frac{d}{dq}f_{2k}(P,Q,R) + Pf_{2k}(P,Q,R).$$

Thus, in the notation (14.2.4),

$$f_{2k+2}(P,Q,R) = 24q\frac{d}{dq}f_{2k}(P,Q,R) + Pf_{2k}(P,Q,R). \tag{14.2.12}$$

With the use of (14.2.12) and the differential equations (14.2.2), it should now be clear how to prove the remaining identities, (iv)–(vi), and so we omit further details. □

Proof of Corollary 14.2.1. By expanding the summands of $P(q)$ in (14.1.2) in geometric series and collecting the coefficients of q^n for each positive integer n, we find that

$$P(q) = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n = -24\sum_{n=0}^{\infty}\sigma(n)q^n,$$

upon using the definition $\sigma(0) = -\frac{1}{24}$. Thus, by (14.2.1), Entry 14.2.1 (i) can be written in the form

$$\begin{aligned}
-24\sum_{j=0}^{\infty}\sigma(j)q^j \cdot &\left(1+\sum_{k=1}^{\infty}(-1)^k\left\{q^{k(3k-1)/2}+q^{k(3k+1)/2}\right\}\right)\\
&= 1+\sum_{n=1}^{\infty}(-1)^n\left\{(6n-1)^2q^{n(3n-1)/2}+(6n+1)^2q^{n(3n+1)/2}\right\}.
\end{aligned} \tag{14.2.13}$$

Equating coefficients of $q^n, n \geq 1$, on both sides of (14.2.13), we complete the proof. □

Proof of Entry 14.2.2. We apply induction on k. For $k = 1, 2$, the assertion (14.2.5) is true by Entry 14.2.1(i), (ii). Assume therefore that (14.2.5) is valid; we shall prove (14.2.5) for k replaced by $k+1$. Our proof employs (14.2.12).

The terms involving P^{k-6}, which are not displayed on the right side of (14.2.5), are of the forms $c_1P^{k-6}R^2$, $c_2P^{k-6}Q^3$, and $c_3P^{k-6}RQ^2$, for certain constants c_1, c_2, and c_3. If we differentiate each of these expressions and use the differential equations (14.2.2), we can easily check that no terms like the five displayed forms in (14.2.5) arise. Thus, when applying (14.2.12) along with induction on k, we need only concern ourselves with the derivatives of the five displayed terms in (14.2.5); no further contributions are made by the derivatives of undisplayed terms to the five coefficients with k replaced by $k+1$.

By (14.2.12), (14.2.2), and induction, we find that

$$\begin{aligned}
f_{2k+2}(P,Q,R) = 1\cdot 3\cdots(2k-1)\Bigg\{&kP^{k-1}\cdot 2(P^2-Q)\\
&-\frac{k(k-1)(k-2)}{3}P^{k-3}Q\cdot 2(P^2-Q)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{k(k-1)}{3} P^{k-2} \cdot 8(PQ - R) \\
&+ \frac{8k(k-1)(k-2)(k-3)}{45} P^{k-4} R \cdot 2(P^2 - Q) \\
&+ \frac{8k(k-1)(k-2)}{45} P^{k-3} \cdot 12(PR - Q^2) \\
&- \frac{11k(k-1)(k-2)(k-3)(k-4)}{210} P^{k-5} Q^2 \cdot 2(P^2 - Q) \\
&- \frac{11k(k-1)(k-2)(k-3)}{210} P^{k-4} \cdot 2Q \cdot 8(PQ - R) \\
&+ \frac{152k(k-1)(k-2)(k-3)(k-4)(k-5)}{14175} \\
&\quad \times P^{k-6} QR \cdot 2(P^2 - Q) \\
&+ \frac{152k(k-1)(k-2)(k-3)(k-4)}{14175} P^{k-5} R \cdot 8(PQ - R) \\
&+ \frac{152k(k-1)(k-2)(k-3)(k-4)}{14175} P^{k-5} Q \\
&\quad \times 12(PR - Q^2) + \cdots \Bigg\} \\
&+ 1 \cdot 3 \cdots (2k-1) \Bigg\{ P^{k+1} - \frac{k(k-1)}{3} P^{k-1} Q \\
&+ \frac{8k(k-1)(k-2)}{45} P^{k-2} R \\
&- \frac{11k(k-1)(k-2)(k-3)}{210} P^{k-3} Q^2 \\
&+ \frac{152k(k-1)(k-2)(k-3)(k-4)}{14175} P^{k-4} QR + \cdots \Bigg\}.
\end{aligned}
$$

The remaining task is to collect coefficients of the five terms P^{k+1}, $P^{k-1}Q$, $P^{k-2}R$, $P^{k-3}Q^2$, and $P^{k-4}QR$. Upon completing this routine, but admittedly tedious, task, we complete the proof of the entry as stated by Ramanujan in [244]. □

Beginning with his paper [240] and notebooks [243], Ramanujan devoted considerable attention to Eisenstein series, most notably to P, Q, and R, defined by (14.1.2)–(14.1.4). In particular, see [53, pp. 318–333], [54, Chapters 16, 17], and [57, Chapter 33]. The identities in [54, pp. 59, 61–65] are particularly related to the ones proved above.

The functions Q and R can be represented or evaluated in terms of parameters prominent in the the theory of elliptic functions [54, pp. 126–127]. The function P does have one representation in terms of elliptic function parameters [54, p. 120, Entry 9 (iv)], but it is in terms of dz/dx, where $z := z(x) := {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)$, and where $q := \exp(-\pi\, z(1-x)/z(x))$. Evaluations of Q and R can be given in terms of z and x; dz/dx does not appear.

Perhaps the representation of P given in Entry 14.2.1(i) will prove to be more useful than the aforementioned representation for P.

Besides Corollary 14.2.1, other identities of Ramanujan can be reformulated in terms of divisor sums $\sigma_k(n) := \sum_{d|n} d^k$. In particular, see [53, pp. 326–329] and the references cited there. By far, the most comprehensive study of identities of this sort has been undertaken by J.G. Huard, Z.M. Ou, B.K. Spearman, and K.S. Williams [181], where many references to the literature can also be found. On the other hand, R.A. Rankin [245] used elementary identities for divisor sums to establish relations between Eisenstein series. In particular, he proved Ramanujan's differential equations (14.2.2) along these lines.

14.3 The Series $U_n(q)$

Recall that the series $U_n(q)$ is defined in (14.1.5). As in the previous section, we use Ramanujan's differential equations (14.2.2) and (14.2.7).

The key to Ramanujan's work on $U_n(q)$ is the following differential-recurrence relation [244, p. 369].

Entry 14.3.1 (p. 369). *For each nonnegative integer n,*

$$U_{n+2}(q) = P(q)U_n(q) + 8qU_n'(q). \tag{14.3.1}$$

Proof. By the definition of $U_n(q)$ in (14.1.5),

$$U_n'(q) = \frac{F_n'(q)(q;q)_\infty - 3F_n(q)\dfrac{d}{dq}(q;q)_\infty}{(q;q)_\infty^4},$$

so that, by (14.2.7),

$$\begin{aligned} P(q)U_n(q) + 8qU_n'(q) &= \left(1 + 24q\frac{\dfrac{d}{dq}(q;q)_\infty}{(q;q)_\infty}\right)\frac{F_n(q)}{(q;q)_\infty^3} \\ &\quad + \frac{8qF_n'(q)(q;q)_\infty - 24F_n(q)q\dfrac{d}{dq}(q;q)_\infty}{(q;q)_\infty^4} \\ &= \frac{F_n(q) + 8qF_n'(q)}{(q;q)_\infty^3}. \end{aligned} \tag{14.3.2}$$

On the other hand, by a simple calculation,

$$\begin{aligned} 8qF_n'(q) &= \sum_{j=1}^\infty (-1)^{j-1}(2j-1)^{n+1}\left((4j^2-4j+1)-1\right)q^{j(j-1)/2} \\ &= F_{n+2}(q) - F_n(q). \end{aligned} \tag{14.3.3}$$

Substituting (14.3.3) into (14.3.2) and simplifying, we complete the proof. □

Entry 14.3.2 (p. 369). *If* $U_n(q)$ *is defined by* (14.1.5), *then*

$$U_0(q) = 1, \tag{14.3.4}$$
$$U_2(q) = P, \tag{14.3.5}$$
$$U_4(q) = \frac{1}{3}\left(5P^2 - 2Q\right), \tag{14.3.6}$$
$$U_6(q) = \frac{1}{9}\left(35P^3 - 42PQ + 16R\right), \tag{14.3.7}$$
$$U_8(q) = \frac{1}{3}\left(35P^4 - 84P^2Q - 12Q^2 + 64PR\right), \tag{14.3.8}$$
$$U_{10}(q) = \frac{1}{9}\left(385P^5 - 1540P^3Q - 660PQ^2 + 1760P^2R + 64QR\right). \tag{14.3.9}$$

Proof. The trivial equality (14.3.4) follows immediately from (14.1.5) and Jacobi's identity (14.1.6).

Setting $n = 0$ in (14.3.1) and using (14.3.4), we deduce (14.3.5).

Next, setting $n = 2$ in (14.3.1), employing (14.3.5), and then using the first equation in (14.2.2), we easily complete the proof of (14.3.6).

Fourthly, apply the differential operator $q\frac{d}{dq}$ to (14.3.6), use (14.3.1), and then employ the first two equations of (14.2.2) to find that

$$U_6 - PU_4 = \frac{40}{3} \cdot 2P\left(\frac{P^2 - Q}{12}\right) - \frac{16}{3}\frac{PQ - R}{3}.$$

The desired result (14.3.7) now follows from (14.3.6) and simplification.

Fifthly, apply the differential operator $q\frac{d}{dq}$ to (14.3.7), use (14.3.1), and then employ all the equations of (14.2.2) to find that

$$U_8 - PU_6 = \frac{8}{9}\left(105P^2\frac{P^2 - Q}{12} - 42Q\frac{P^2 - Q}{12} - 42P\frac{PQ - R}{3} + 16\frac{PR - Q^2}{2}\right).$$

If we use (14.3.7) on the left side above, collect terms with like powers, and simplify, we obtain (14.3.8).

Lastly, apply the differential operator $q\frac{d}{dq}$ to (14.3.8), use (14.3.1), and then employ all the equations of (14.2.2) to find that

$$U_{10} - PU_8 = \frac{8}{3}\left(140P^3\frac{P^2 - Q}{12} - 168PQ\frac{P^2 - Q}{12} - 84P^2\frac{PQ - R}{3}\right.$$
$$\left.-24Q\frac{PQ - R}{3} + 64R\frac{P^2 - Q}{12} + 64P\frac{PR - Q^2}{2}\right).$$

Using (14.3.8) on the left side above and then simplifying, we arrive at (14.3.9) to complete the proof. □

It is easy to see from our calculations above that we can deduce the following general theorem stated by Ramanujan [244, p. 369].

Entry 14.3.3 (p. 369). *For any positive integer s,*

$$U_{2s} = \sum K_{\ell,m,n} P^{\ell} Q^{m} R^{n}, \tag{14.3.10}$$

where the sum is over all nonnegative triples of integers ℓ, m, n such that $\ell + 2m + 3n = s$.

Although one can find formulas for some of the coefficients $K_{\ell,m,n}$ in (14.3.10), it seems extremely difficult to find a general formula for all $K_{\ell,m,n}$.

The identity (14.3.5) arose in a proof of Berndt, S.H. Chan, Liu, and H. Yesilyurt [70] of the identity

$$1 + 3\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} - 27\sum_{n=1}^{\infty} \frac{nq^{9n}}{1-q^{9n}} = \frac{(q^3;q^3)_\infty^{10}}{(q;q)_\infty^3 (q^9;q^9)_\infty^3},$$

which was used by these four authors to establish a new identity for $(q;q)_\infty^{10}$.

The proofs in Section 14.2 first appeared in a paper by Berndt and A.J. Yee [77], while those in Section 14.3 are taken from [70]. Also see the monograph by K. Venkatachaliengar [272, pp. 31–32]. Another approach to the identities proved in this chapter has been devised by H.H. Chan [102]. These ideas of Ramanujan have been extended in a beautiful way by Chan, S. Cooper, and P.C. Toh [105], [106]. They have found infinite classes of identities wherein the roles of $(q;q)_\infty$ and $(q;q)_\infty^3$ are replaced by $(q;q)_\infty^n$, $n = 2, 4, 6, 8, 10, 14, 26$. In other words, Chan, Cooper, and Toh evaluate certain classes of infinite series in terms of one of the three aforementioned powers times a polynomial in the Eisenstein series P, Q, and R. Z.-G. Liu [210] has also found beautiful expansions for $(q;q)_\infty^n$ for $n = 2, 6, 8, 10$. An equivalent formulation of the forgoing identities for $n = 8$ has also been derived by Z. Cao [97]. On the other hand, H. Hahn [171] has established an analogue of Entry 14.2.1 involving Eisenstein series on $\Gamma_0(2)$. Further generalizations can be found in T. Huber's doctoral dissertation [182, Chapter 4].

In closing this chapter, we remark that recently there have been several new approaches to Ramanujan's differential equations (14.1.2)–(14.1.4), with many providing connections with Riccati differential equations and other differential equations. For example, see papers by Chan [102], J.M. Hill, Berndt, and Huber [180], Huber [183], P. Guha and D. Mayer [162], Hahn [171], M.J. Ablowitz, S. Chakravarty, and Hahn [1], and R.S. Maier [219].

15

Eisenstein Series and Approximations to π

15.1 Introduction

On page 211 in his lost notebook, in the pagination of [244], Ramanujan listed eight integers, 11, 19, 27, 43, 67, 163, 35, and 51 at the left margin. To the right of each integer, Ramanujan recorded a linear equation in Q^3 and R^2. Although Ramanujan did not indicate the definitions of Q and R, we can easily (and correctly) ascertain that Q and R are the Eisenstein series

$$Q(q) := 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}$$

and

$$R(q) := 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n},$$

where $|q| < 1$. To the right of each equation in Q^3 and R^2, Ramanujan entered an equality involving π and square roots. (For the integer 51, the linear equation and the equality involving π are not in fact, recorded by Ramanujan.)

The equations in Q^3 and R^2 cannot possibly hold for all values of q with $|q| < 1$. Thus, the first task was to find the correct value of q for each equation. After trial and error with the aid of S.H. Son, we found that $q = -\exp(-\pi\sqrt{n})$, where n is the integer at the left margin. (We later read that K. Venkatachaliengar [272, p. 135] had also discovered that $q = -\exp(-\pi\sqrt{n})$.) The equalities in the third column lead to approximations to π that are reminiscent of approximations given by Ramanujan in his famous paper on modular equations and approximations to π [239], [242, p. 33] and studied extensively by J.M. and P.B. Borwein [86, Chapter 5]. This page in the lost notebook is also closely connected with theorems connected with the modular j-invariant stated by Ramanujan on the last two pages of his third notebook [244] and proved by Berndt and H.H. Chan [63], [57, pp. 309–322].

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In Section 15.2, we prove a very simple general theorem from which the linear equations in Q^3 and R^2 in the second column follow as corollaries. In Sections 15.3 and 15.4, we offer two methods for proving the equalities in the third column and show how they lead to approximations to π. In Section 15.6, we prove a general series formula for $1/\pi$ that is equivalent to a formula found by D.V. and G.V. Chudnovsky [132] and the Borweins [91]. The first series representations for $1/\pi$ of this type were found by Ramanujan [239], [242, pp. 23–39] and first proved in print by the Borweins [86], [88]. Three of the series from [239] are found on page 370 in the lost notebook [244]. We use Ramanujan's ideas that are briefly sketched in [239], [242, pp. 23–39] to establish these three series representations for $1/\pi$. One of Ramanujan's series for $1/\pi$ from [239] yields 8 digits of π per term, while one of the Borweins [87] gives 50 digits of π per term. The method of Berndt and Chan gives a series for $1/\pi$ that yields about 73 or 74 digits of π per term [64].

15.2 Eisenstein Series and the Modular j-Invariant

Recall the definition of the modular j-invariant $j(\tau)$,

$$j(\tau) = 1728\frac{Q^3(q)}{Q^3(q) - R^2(q)}, \qquad q = e^{2\pi i\tau}, \qquad \operatorname{Im} \tau > 0. \tag{15.2.1}$$

In particular, if n is a positive integer,

$$j\left(\frac{3+\sqrt{-n}}{2}\right) = 1728\frac{Q_n^3}{Q_n^3 - R_n^2}, \tag{15.2.2}$$

where, for brevity, we set

$$Q_n := Q(-e^{-\pi\sqrt{n}}) \qquad \text{and} \qquad R_n := R(-e^{-\pi\sqrt{n}}). \tag{15.2.3}$$

In his third notebook, at the top of page 392 in the pagination of [244], Ramanujan defined a certain function J_n of singular moduli, which, as Berndt and Chan [63] easily showed, has the representation

$$J_n = -\frac{1}{32}\sqrt[3]{j\left(\frac{3+\sqrt{-n}}{2}\right)}. \tag{15.2.4}$$

Hence, from (15.2.2) and (15.2.4),

$$(-32J_n)^3 = 1728\frac{Q_n^3}{Q_n^3 - R_n^2}. \tag{15.2.5}$$

After a simple manipulation of (15.2.5), we deduce the following theorem.

Theorem 15.2.1. *For each positive integer n,*

$$\left(\left(\frac{8}{3}J_n\right)^3+1\right)Q_n^3-\left(\frac{8}{3}J_n\right)^3R_n^2=0, \tag{15.2.6}$$

where J_n is defined by (15.2.4)*, and Q_n and R_n are defined by* (15.2.3)

Entry 15.2.1 (p. 211). *We have*

$$\begin{aligned}
539Q_{11}^3-512R_{11}^2&=0,\\
(8^3+1)Q_{19}^3-8^3R_{19}^2&=0,\\
(40^3+9)Q_{27}^3-40^3R_{27}^2&=0,\\
(80^3+1)Q_{43}^3-80^3R_{43}^2&=0,\\
(440^3+1)Q_{67}^3-440^3R_{67}^2&=0,\\
(53360^3+1)Q_{163}^3-53360^3R_{163}^2&=0,\\
((60+28\sqrt{5})^3+27)Q_{35}^3-(60+28\sqrt{5})^3R_{35}^2&=0,
\end{aligned}$$

and

$$((4(4+\sqrt{17})^{2/3}(5+\sqrt{17}))^3+1)Q_{51}^3-(4(4+\sqrt{17})^{2/3}(5+\sqrt{17}))^3R_{51}^2=0.$$

Proof. In [63], [57, pp. 310–311], it was shown that

$$\begin{aligned}
J_{11}&=1, & J_{19}&=3,\\
J_{27}&=5\cdot 3^{1/3}, & J_{43}&=30,\\
J_{67}&=165, & J_{163}&=20,010,\\
J_{35}&=\sqrt{5}\left(\frac{1+\sqrt{5}}{2}\right)^4, & J_{51}&=3(4+\sqrt{17})^{2/3}\left(\frac{5+\sqrt{17}}{2}\right).
\end{aligned} \tag{15.2.7}$$

Using (15.2.7) in (15.2.6), we readily deduce all eight equations in Q_n and R_n. □

15.3 Eisenstein Series and Equations in π: First Method

Recall that

$$P(q):=1-24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}, \qquad |q|<1, \tag{15.3.1}$$

and put

$$P_n:=P(-e^{-\pi\sqrt{n}}). \tag{15.3.2}$$

Next, set

$$j_n = j\left(\frac{3+\sqrt{-n}}{2}\right),$$

$$b_n = \{n(1728 - j_n)\}^{1/2}, \tag{15.3.3}$$

and

$$a_n = \frac{1}{6}b_n\left\{1 - \frac{Q_n}{R_n}\left(P_n - \frac{6}{\pi\sqrt{n}}\right)\right\}. \tag{15.3.4}$$

The numbers a_n and b_n arise in series representations for $1/\pi$ proved by the Chudnovskys [132] and the Borweins [88], namely,

$$\frac{1}{\pi} = \frac{1}{\sqrt{-j_n}}\sum_{k=0}^{\infty}\frac{(6k)!}{(3k)!(k!)^3}\frac{a_n + kb_n}{j_n^k}, \tag{15.3.5}$$

where $(c)_0 = 1, (c)_k = c(c+1)\cdots(c+k-1),$ for $k \geq 1$. These authors have calculated a_n and b_n for several values of n. We show how (15.3.3) and (15.3.4) lead to a formula from which Ramanujan's equalities in the third column on page 211 follow.

From (15.2.6), we easily see that

$$\frac{Q_n}{R_n} = \frac{1}{\sqrt{Q_n}}\left(\frac{\left(\frac{8}{3}J_n\right)^3 + 1}{\left(\frac{8}{3}J_n\right)^3}\right)^{-1/2}, \tag{15.3.6}$$

and from (15.3.4), we find that

$$\frac{Q_n}{R_n}\left(\frac{6}{\pi} - \sqrt{n}P_n\right) = 6\sqrt{n}\frac{a_n}{b_n} - \sqrt{n}. \tag{15.3.7}$$

The substitution of (15.3.6) into (15.3.7) leads to the following theorem.

Theorem 15.3.1. *If $P_n, b_n, a_n,$ and J_n are defined by* (15.3.2)–(15.3.4) *and* (15.2.4), *respectively, then*

$$\frac{1}{\sqrt{Q_n}}\left(\sqrt{n}P_n - \frac{6}{\pi}\right) = \sqrt{n}\left(1 - 6\frac{a_n}{b_n}\right)\left(\frac{\left(\frac{8}{3}J_n\right)^3 + 1}{\left(\frac{8}{3}J_n\right)^3}\right)^{1/2}. \tag{15.3.8}$$

Entry 15.3.1 (p. 211). *We have*

$$\frac{1}{\sqrt{Q_{11}}}\left(\sqrt{11}P_{11} - \frac{6}{\pi}\right) = \sqrt{2},$$

$$\frac{1}{\sqrt{Q_{19}}}\left(\sqrt{19}P_{19} - \frac{6}{\pi}\right) = \sqrt{6},$$

$$\frac{1}{\sqrt{Q_{27}}}\left(\sqrt{27}P_{27} - \frac{6}{\pi}\right) = 3\sqrt{\frac{6}{5}},$$

$$\frac{1}{\sqrt{Q_{43}}}\left(\sqrt{43}P_{43}-\frac{6}{\pi}\right)=6\sqrt{\frac{3}{5}},$$
$$\frac{1}{\sqrt{Q_{67}}}\left(\sqrt{67}P_{67}-\frac{6}{\pi}\right)=19\sqrt{\frac{6}{55}},$$
$$\frac{1}{\sqrt{Q_{163}}}\left(\sqrt{163}P_{163}-\frac{6}{\pi}\right)=362\sqrt{\frac{3}{3335}},$$
$$\frac{1}{\sqrt{Q_{35}}}\left(\sqrt{35}P_{35}-\frac{6}{\pi}\right)=(2+\sqrt{5})\sqrt{\frac{2}{\sqrt{5}}},$$
$$\frac{1}{\sqrt{Q_{51}}}\left(\sqrt{51}P_{51}-\frac{6}{\pi}\right)= .$$

Ramanujan's formulation of the first example of Entry 15.3.1 is apparently given by

$$\frac{\sqrt{11}-\dfrac{6}{\pi}+\cdots}{\sqrt{1-240\left(\dfrac{1^3}{e^{\pi\sqrt{11}}\cdots}\right)}}=\sqrt{2}.$$

(The denominator with 1^3 in the numerator is unreadable.) Further equalities are even briefer, with $\sqrt{Q_n}$ replaced by $\sqrt{\cdot}$. Note that P_n is replaced by "$1+\cdots$" in Ramanujan's examples. Also observe that Ramanujan did not record the right side when $n=51$. Because it is unwieldy, we also have not recorded it. However, readers can readily complete the equality, since J_{51} is given in (15.2.7), and a_{51} and b_{51} are given in the next table.

Proof. The first six values of a_n and b_n were calculated by the Borweins [88, pp. 371–372]. The values for $n=35$ and 51 were calculated by Berndt and Chan [64]. We record all eight pairs of values for a_n and b_n in the following table:

n	a_n	b_n
11	60	616
19	300	4104
27	1116	18216
43	9468	195048
67	122124	3140424
163	163096908	6541681608
35	$1740+768\sqrt{5}$	$32200+14336\sqrt{5}$
51	$11820+2880\sqrt{17}$	$265608+64512\sqrt{17}$

If we substitute these values of a_n and b_n in Theorem 15.3.1, we obtain, after some calculation and simplification, Ramanujan's equalities. ∎

Theorem 15.3.1 and the last set of examples yield approximations to π. Let r_n denote the right-hand side of (15.3.8). If we use the expansions

$$P_n = 1 + 24e^{-\pi\sqrt{n}} - \cdots \qquad \text{and} \qquad \sqrt{Q_n} = 1 - 120e^{-\pi\sqrt{n}} + \cdots,$$

we easily find that

$$\pi = \frac{6}{\sqrt{n} - r_n}\left(1 - \frac{24\sqrt{n} + 120r_n}{\sqrt{n} - r_n}e^{-\pi\sqrt{n}} + \cdots\right).$$

We thus have proved the following theorem.

Theorem 15.3.2. *We have*

$$\pi \approx \frac{6}{\sqrt{n} - r_n} =: A_n,$$

with the error approximately equal to

$$144\frac{\sqrt{n} + 5r_n}{(\sqrt{n} - r_n)^2}e^{-\pi\sqrt{n}},$$

where r_n denotes the right-hand side of (15.3.8).

See Ramanujan's paper [239], [242, p. 33] for other approximations to π of this sort.

In the table below, we record the decimal expansion of each approximation A_n and the number N_n of digits of π agreeing with the approximation.

n	A_n	N_n
11	3.1538...	1
19	3.1423...	2
27	3.1416621...	3
43	3.141593...	5
67	3.14159266...	7
163	3.14159265358980...	12
35	3.141601...	3
51	3.14159289...	6

15.4 Eisenstein Series and Equations in π: Second Method

Set $\mathbf{P} := \mathbf{P}(q) := P(-q)$, $\mathbf{Q} := \mathbf{Q}(q) := Q(-q)$, $\mathbf{R} := \mathbf{R}(q) := R(-q)$, $\boldsymbol{\Delta} := \boldsymbol{\Delta}(q) := \mathbf{Q}^3(q) - \mathbf{R}^2(q)$, and $\mathbf{J} := \mathbf{J}(q) := 1728/j\left(\frac{3+\tau}{2}\right)$, where $q = e^{2\pi i\tau}$. Set

$$z^4 := \mathbf{Q} = \left(\frac{\boldsymbol{\Delta}}{\mathbf{J}}\right)^{1/3}, \tag{15.4.1}$$

by (15.2.1). Then, by (15.4.1) and the definition of $\mathbf{\Delta}$,

$$\mathbf{R} = \sqrt{\mathbf{Q}^3 - \mathbf{\Delta}} = \sqrt{\frac{\mathbf{\Delta}}{\mathbf{J}}}\sqrt{1-\mathbf{J}} = z^6\sqrt{1-\mathbf{J}}. \tag{15.4.2}$$

Recall the differential equations [240], [242, p. 142]

$$q\frac{dP}{dq} = \frac{P^2(q) - Q(q)}{12}, \quad q\frac{dQ}{dq} = \frac{P(q)Q(q) - R(q)}{3}, \quad q\frac{dR}{dq} = \frac{P(q)R(q) - Q^2(q)}{2}, \tag{15.4.3}$$

which yield the associated differential equations

$$q\frac{d\mathbf{P}}{dq} = \frac{\mathbf{P}^2(q) - \mathbf{Q}(q)}{12}, \quad q\frac{d\mathbf{Q}}{dq} = \frac{\mathbf{P}(q)\mathbf{Q}(q) - \mathbf{R}(q)}{3}, \quad q\frac{d\mathbf{R}}{dq} = \frac{\mathbf{P}(q)\mathbf{R}(q) - \mathbf{Q}^2(q)}{2}. \tag{15.4.4}$$

Now, by rearranging the second equation in (15.4.4), with the help of (15.4.1) and (15.4.2), we find that

$$\mathbf{P}(q) = \frac{\mathbf{R}(q)}{\mathbf{Q}(q)} + \frac{12q}{z}\frac{dz}{dq}. \tag{15.4.5}$$

From the chain rule and (15.4.5), it follows that, for any positive integer n,

$$\mathbf{P}(q^n) = \frac{\mathbf{R}(q^n)}{\mathbf{Q}(q^n)} + \frac{12q}{nz(q^n)}\frac{dz(q^n)}{dq}.$$

Subtracting (15.4.5) from the last equality and setting

$$m := \frac{z(q)}{z(q^n)},$$

we find that

$$\begin{aligned} n\mathbf{P}(q^n) - \mathbf{P}(q) &= n\frac{\mathbf{R}(q^n)}{\mathbf{Q}(q^n)} - \frac{\mathbf{R}(q)}{\mathbf{Q}(q)} + 12\frac{q}{z(q^n)}\frac{dz(q^n)}{dq} - 12\frac{q}{z(q)}\frac{dz(q)}{dq} \\ &= n\frac{\mathbf{R}(q^n)}{\mathbf{Q}(q^n)} - \frac{\mathbf{R}(q)}{\mathbf{Q}(q)} - 12\frac{q}{m}\frac{dm}{dq}. \end{aligned} \tag{15.4.6}$$

Our next aim is to replace $\frac{dm}{dq}$ in (15.4.6) by $\frac{dm}{d\mathbf{J}}(\mathbf{J}(q), \mathbf{J}(q^n))$. From (15.2.1), the definition of $\mathbf{J}$, (15.4.4), (15.4.1), and (15.4.2), upon differentiation, we find that

$$\begin{aligned} q\frac{d\mathbf{J}}{dq} &= \frac{(3\mathbf{Q}^2\mathbf{Q}' - 2\mathbf{R}\mathbf{R}')\mathbf{Q}^3 - 3\mathbf{Q}^2\mathbf{Q}'(\mathbf{Q}^3 - \mathbf{R}^2)}{\mathbf{Q}^6} \\ &= \frac{\{\mathbf{Q}^2(\mathbf{P}\mathbf{Q} - \mathbf{R}) - \mathbf{R}(\mathbf{P}\mathbf{R} - \mathbf{Q}^2)\}\mathbf{Q}^3 - \mathbf{Q}^2(\mathbf{P}\mathbf{Q} - \mathbf{R})(\mathbf{Q}^3 - \mathbf{R}^2)}{\mathbf{Q}^6} \\ &= \frac{\mathbf{R}\mathbf{Q}^3 - \mathbf{R}^3}{\mathbf{Q}^4} = \frac{\mathbf{R}\mathbf{\Delta}}{\mathbf{Q}^4} = z^6\sqrt{1-\mathbf{J}}\frac{\mathbf{J}}{\mathbf{Q}} = z^2\mathbf{J}\sqrt{1-\mathbf{J}}, \end{aligned} \tag{15.4.7}$$

which implies that

$$z^2(q) = \frac{1}{\mathbf{J}(q)\sqrt{1-\mathbf{J}(q)}} q \frac{d\mathbf{J}(q)}{dq}. \tag{15.4.8}$$

Replacing q by q^n in (15.4.8) and simplifying, we deduce that

$$z^2(q^n) = \frac{1}{n\mathbf{J}(q^n)\sqrt{1-\mathbf{J}(q^n)}} q \frac{d\mathbf{J}(q^n)}{dq}. \tag{15.4.9}$$

Using (15.4.8) and (15.4.9), we conclude that

$$m^2 = n\frac{\mathbf{J}(q^n)\sqrt{1-\mathbf{J}(q^n)}}{\mathbf{J}(q)\sqrt{1-\mathbf{J}(q)}} \frac{d\mathbf{J}(q)}{d\mathbf{J}(q^n)}. \tag{15.4.10}$$

It is well known that there is a relation (known as the class equation) between $j(\tau)$ and $j(n\tau)$ for any integer n [146, p. 231, Theorem 11.18(i)]. With the definition of $\mathbf{J}$ given at the beginning of this section, the class equation translates to a relation between $\mathbf{J}(q)$ and $\mathbf{J}(q^n)$. It follows that

$$\frac{d\mathbf{J}(q)}{d\mathbf{J}(q^n)} = F(\mathbf{J}(q), \mathbf{J}(q^n)), \tag{15.4.11}$$

for some rational function $F(x, y)$. Thus, by (15.4.10) and (15.4.11), we may differentiate m with respect to $\mathbf{J}$, and so, by (15.4.7) and the definition of $m(q)$,

$$\begin{aligned} 2\frac{q}{m(q)}\frac{dm}{dq} &= 2z(q)z(q^n)\frac{q}{z^2(q)}\frac{dm}{dq} \\ &= 2z^2(q^n)m(q)\frac{dm}{d\mathbf{J}}q\frac{d\mathbf{J}}{dq} = z^2(q^n)\mathbf{J}\sqrt{1-\mathbf{J}}\frac{dm^2(q)}{d\mathbf{J}}. \end{aligned}$$

Using this in (15.4.6), we deduce that

$$\frac{n\mathbf{P}(q^n) - \mathbf{P}(q)}{z(q)z(q^n)} = n\frac{\mathbf{R}(q^n)}{\mathbf{Q}(q^n)} - \frac{\mathbf{R}(q)}{\mathbf{Q}(q)} - 6z^2(q^n)\mathbf{J}(q)\sqrt{1-\mathbf{J}(q)}\frac{dm^2}{d\mathbf{J}}. \tag{15.4.12}$$

If we put $q = e^{-\pi/\sqrt{n}}, n > 0$, (15.4.12) takes the shape

$$\begin{aligned} n\mathbf{P}(e^{-\pi\sqrt{n}}) - \mathbf{P}(e^{-\pi/\sqrt{n}}) &= n\frac{\mathbf{R}(e^{-\pi\sqrt{n}})}{\mathbf{Q}(e^{-\pi\sqrt{n}})} - \frac{\mathbf{R}(e^{-\pi/\sqrt{n}})}{\mathbf{Q}(e^{-\pi/\sqrt{n}})} \\ &\quad - 6z^2(e^{-\pi\sqrt{n}})\mathbf{J}(e^{-\pi/\sqrt{n}})\sqrt{1-\mathbf{J}(e^{-\pi/\sqrt{n}})} \\ &\quad \times \frac{dm^2}{d\mathbf{J}}\left(\mathbf{J}(e^{-\pi\sqrt{n}}), \mathbf{J}(e^{-\pi/\sqrt{n}})\right). \end{aligned} \tag{15.4.13}$$

It is well known that [120, p. 84]

$$\mathbf{J}(e^{-\pi/\sqrt{n}}) = \mathbf{J}(e^{-\pi\sqrt{n}}). \tag{15.4.14}$$

Furthermore, if

$$\varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2}, \qquad |q| < 1, \tag{15.4.15}$$

then [54, p. 127, Entries 13(iii), (iv)]

$$Q(q) = z_2^4(1 + 14x_2 + x_2^2) \tag{15.4.16}$$

and

$$R(q) = z_2^6(1 + x_2)(1 - 34x_2 + x_2^2), \tag{15.4.17}$$

where [54, pp. 122–123, Entries 10(i), 11(iii)]

$$z_2 := \varphi^2(q) \quad \text{and} \quad x_2 := 16q\frac{\psi^4(q^2)}{\varphi^4(q)}. \tag{15.4.18}$$

Replacing q by $-q$ in (15.4.16) and (15.4.17), and using (15.4.18), we find that

$$\mathbf{Q}(q) = \varphi^8(-q) - 224q\varphi^4(-q)\psi^4(q^2) + 16^2q^2\psi^8(q^2) \tag{15.4.19}$$

and

$$\begin{aligned}\mathbf{R}(q) = {}&(\varphi^4(-q) - 16q\psi^4(q^2)) \\ &\times (\varphi^8(-q) + 544q\varphi^4(-q)\psi^4(q^2) + 16^2q^2\psi^8(q^2)).\end{aligned} \tag{15.4.20}$$

Using the transformation formula [54, p. 43, Entry 27(ii)]

$$\varphi(e^{-\pi/t}) = 2e^{-\pi t/4}\sqrt{t}\psi(e^{-2\pi t})$$

in (15.4.19) and (15.4.20), we deduce that

$$\mathbf{Q}(e^{-\pi/\sqrt{n}}) = n^2\mathbf{Q}(e^{-\pi\sqrt{n}}) \tag{15.4.21}$$

and

$$\mathbf{R}(e^{-\pi/\sqrt{n}}) = -n^3\mathbf{R}(e^{-\pi\sqrt{n}}). \tag{15.4.22}$$

Using (15.4.14), (15.4.22), and (15.4.21), we may rewrite (15.4.13) as

$$\begin{aligned}&n\mathbf{P}(e^{-\pi\sqrt{n}}) - \mathbf{P}(e^{-\pi/\sqrt{n}}) \\ &\quad = 2n\frac{\mathbf{R}(e^{-\pi\sqrt{n}})}{\mathbf{Q}(e^{-\pi\sqrt{n}})} - 6z^2(e^{-\pi\sqrt{n}})\mathbf{J}_n\sqrt{1 - \mathbf{J}_{1/n}}\frac{dm^2}{d\mathbf{J}}(\mathbf{J}_n, \mathbf{J}_n)\end{aligned}$$

$$= \left(2n\sqrt{1-\mathbf{J}_n} - 6\mathbf{J}_n\sqrt{1-\mathbf{J}_{1/n}}\frac{dm^2}{d\mathbf{J}}\left(\mathbf{J}_n, \mathbf{J}_n\right) \right) z^2(e^{-\pi\sqrt{n}}), \tag{15.4.23}$$

where

$$\mathbf{J}_k = \mathbf{J}(e^{-\pi\sqrt{k}}), \qquad k > 0. \tag{15.4.24}$$

This gives the first relation between $\mathbf{P}(e^{-\pi\sqrt{n}})$ and $\mathbf{P}(e^{-\pi/\sqrt{n}})$.

Recall the definitions of Ramanujan's function $f(-q)$ and the Dedekind eta function $\eta(\tau)$, namely,

$$f(-q) := (q;q)_\infty =: e^{-2\pi i\tau/24}\eta(\tau), \qquad q = e^{2\pi i\tau}, \qquad \operatorname{Im}\tau > 0.$$

The function f satisfies the well-known transformation formula [54, p. 43]

$$n^{1/4}e^{-\pi\sqrt{n}/24}f(e^{-\pi\sqrt{n}}) = e^{-\pi/(24\sqrt{n})}f(e^{-\pi/\sqrt{n}}), \qquad n > 0. \tag{15.4.25}$$

Logarithmically differentiating (15.4.25) with respect to n, multiplying both sides by $48n^{3/2}/\pi$, rearranging terms, and employing the definition of $P(q)$ given in (15.3.1), we find that

$$\frac{12\sqrt{n}}{\pi} = n\mathbf{P}(e^{-\pi\sqrt{n}}) + \mathbf{P}(e^{-\pi/\sqrt{n}}). \tag{15.4.26}$$

This gives a second relation between $\mathbf{P}(e^{-\pi\sqrt{n}})$ and $\mathbf{P}(e^{-\pi/\sqrt{n}})$.

Now adding (15.4.23) and (15.4.26) and dividing by 2, we arrive at

$$n\mathbf{P}(e^{-\pi\sqrt{n}}) = \frac{6\sqrt{n}}{\pi} + \left(n\sqrt{1-\mathbf{J}_n} - 3\mathbf{J}_n\sqrt{1-\mathbf{J}_{1/n}}\frac{dm^2}{d\mathbf{J}}(\mathbf{J}_n, \mathbf{J}_n) \right) z^2(e^{-\pi\sqrt{n}}),$$

or, by (15.4.1),

$$\frac{1}{\sqrt{Q_n}}\left(P_n - \frac{6}{\sqrt{n}\pi} \right) = \sqrt{1-\mathbf{J}_n}\left(1 - 3\mathbf{J}_n\frac{\sqrt{1-\mathbf{J}_{1/n}}}{n\sqrt{1-\mathbf{J}_n}}\frac{dm^2}{d\mathbf{J}}(\mathbf{J}_n, \mathbf{J}_n) \right),$$

where Q_n is defined by (15.2.3), and P_n is defined by (15.3.2). (Be careful: $\mathbf{J}_n \neq J_n$, where J_n is defined by (15.2.4).)

We record the last result in the following theorem, which should be compared with Theorem 15.3.1.

Theorem 15.4.1. *If P_n, Q_n, and $\mathbf{J}_n$ are defined by* (15.3.2), (15.2.3), *and* (15.4.24), *respectively, then*

$$\frac{1}{\sqrt{Q_n}}\left(P_n - \frac{6}{\sqrt{n}\pi} \right) = \sqrt{1-\mathbf{J}_n}\, t_n, \tag{15.4.27}$$

where

$$t_n := \left(1 - 3\mathbf{J}_n\frac{\sqrt{1-\mathbf{J}_{1/n}}}{n\sqrt{1-\mathbf{J}_n}}\frac{dm^2}{d\mathbf{J}}(\mathbf{J}_n, \mathbf{J}_n) \right). \tag{15.4.28}$$

Observe that, by (15.4.1), (15.4.2), and Theorem 15.2.1,

$$\sqrt{1-\mathbf{J}_n} = \left(\frac{\left(\frac{8}{3}J_n\right)^3 + 1}{\left(\frac{8}{3}J_n\right)^3} \right)^{1/2}. \tag{15.4.29}$$

Hence, the values of $\sqrt{1-\mathbf{J}_n}$ for those n given on page 211 of the lost notebook follow immediately from (15.2.7). In order to rederive Entry 15.3.1, it suffices to compute t_n.

Theorem 15.4.2. *If $n > 1$ is an odd positive integer, then t_n lies in the ring class field of $\mathbf{Z}[\sqrt{-n}]$.*

A proof of Theorem 15.4.2 can be found in [64]. In certain cases, one can use Theorem 15.4.2 to empirically calculate t_n; for more details, see [64].

15.5 Page 213

On page 213 in his lost notebook [244], Ramanujan lists three further quotients of Eisenstein series, the second of which is difficult to read. Since Ramanujan did not make any claims or indicate (even cryptically) any associated singular moduli, we have not made any attempt at further investigations. The quotients are given by

$$\frac{1}{31\frac{1}{4}R^2 - 30\frac{1}{4}Q^3},$$
$$\frac{1}{9595\frac{45}{64}R^2 - 9594\frac{45}{64}Q^3},$$
$$\frac{64}{189Q^3 - 125R^2}.$$

Note that if the numerator and denominator of the last quotient is divided by 64, then the difference of the coefficients in each of the three quotients is equal to 1.

15.6 Ramanujan's Series for $1/\pi$

On page 370 in his lost notebook, Ramanujan records three series for $1/\pi$. In fact, these three series are the identities (28)–(30) in Ramanujan's famous paper [239], [242, pp. 36–37]. These series and fourteen further series for $1/\pi$ were established by the Borwein brothers [86, Chapter 5]. In this section, we take a different approach from that of the Borweins and use Eisenstein series to establish a very general series for $1/\pi$ from which the three series on page 370 can be determined.

Throughout this section, we employ the notation of Chapter 13. In particular, we recall the representation for the base q in (13.2.8), the definition of the modular equation of degree n arising from the equation (13.2.5), the definitions of z in (13.2.9) and z_1 and z_n in (13.2.6), and the definition of the multiplier m in (13.2.7). The two most important ingredients in our derivations are Ramanujan's representation for $P(q^2)$ given by [54, p. 120, Entry 9(iv)]

$$P(q^2) = (1-2x)z^2 + 6x(1-x)z\frac{dz}{dx} \tag{15.6.1}$$

and Clausen's formula, which we use in the form [86, p. 180, Theorem 5.7(a)]

$$z^2 = {}_3F_2(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}; 1, 1; X) = \sum_{k=0}^{\infty} A_k X^k, \tag{15.6.2}$$

where

$$A_k := \frac{(\frac{1}{2})_k^3}{k!^3} \qquad \text{and} \qquad X := 4x(1-x). \tag{15.6.3}$$

From (15.6.2) and (15.6.3),

$$2z\frac{dz}{dx} = \sum_{k=0}^{\infty} A_k k X^{k-1} \cdot 4(1-2x). \tag{15.6.4}$$

Hence, from (15.6.1), (15.6.2), (15.6.4), and (15.6.3),

$$\begin{aligned} P(q^2) &= (1-2x)\sum_{k=0}^{\infty} A_k X^k + 3(1-2x)\sum_{k=0}^{\infty} A_k k X^k \\ &= \sum_{k=0}^{\infty} \{(1-2x) + 3(1-2x)k\} A_k X^k. \end{aligned} \tag{15.6.5}$$

We next derive another representation for $P(q^2)$. In the notation

$$q = e^{-y}, \qquad y = \pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)},$$

used by Ramanujan in Chapter 16 and throughout the remainder of his second notebook, we quote from Entry 9(i) in Chapter 17 in his second notebook [243], [54, p. 120]:

$$\frac{dy}{dx} = -\frac{1}{x(1-x)z^2}.$$

A simple application of the chain rule gives

$$\frac{dq}{dx} = \frac{q}{x(1-x)z^2},$$

or

$$q\frac{dx}{dq} = x(1-x)z^2. \tag{15.6.6}$$

Hence, from (15.6.1) and (15.6.6),

$$\frac{6q}{z}\frac{dz}{dq} = P(q^2) - (1-2x)z^2. \tag{15.6.7}$$

If we set $t = q^n$, then from (15.6.7) and the chain rule,

$$\frac{6nq^n}{z(q^n)}\frac{dz}{dt} = n\left(P(q^{2n}) - (1-2x(q^n))z^2(q^n)\right). \tag{15.6.8}$$

Logarithmically differentiating the formula for m in (13.2.7), we deduce that

$$q\frac{m'(q)}{m(q)} = q\frac{z'(q)}{z(q)} - nq^n\frac{z'(q^n)}{z(q^n)}. \tag{15.6.9}$$

Hence, by (15.6.7)–(15.6.9),

$$\begin{aligned} 6q\frac{m'(q)}{m(q)} &= P(q^2) - (1-2x(q))z^2 - nP(q^{2n}) + n(1-2x(q^n))z^2(q^n) \\ &= P(q^2) - nP(q^{2n}) - (1-2x(q))z^2 + n(1-2x(q^n))z^2(q^n). \end{aligned} \tag{15.6.10}$$

Now, set

$$q := e^{-\pi/\sqrt{n}} \qquad \text{and} \qquad x_n := x(e^{-\pi\sqrt{n}}).$$

Then, from (13.2.6),

$$z_n = z(e^{-\pi\sqrt{n}}).$$

Therefore, for this value of q, we rewrite (15.6.10) in the form

$$\begin{aligned} 6e^{-\pi/\sqrt{n}}\frac{m'}{m}(e^{-\pi/\sqrt{n}}) &= P(e^{-2\pi/\sqrt{n}}) - nP(e^{-2\pi\sqrt{n}}) \\ &\quad - (1-2x_{1/n})z_{1/n}^2 + n(1-2x_n)z_n^2. \end{aligned} \tag{15.6.11}$$

Recalling (13.2.8) and the notation $q = e^{-y} = e^{-\pi/\sqrt{n}}$, we see that

$$\frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x_{1/n})}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x_{1/n})} = \frac{1}{\sqrt{n}}. \tag{15.6.12}$$

Also, from the definition of a modular equation revolving around (13.2.5), and from (15.6.12),

$$\frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x_n)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x_n)} = n\frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x_{1/n})}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x_{1/n})} = \sqrt{n}. \tag{15.6.13}$$

Hence, from (15.6.12), (15.6.13), and (13.2.7), we conclude that

$$1 - x_n = x_{1/n}, \qquad z_{1/n} = \sqrt{n} z_n, \qquad \text{and} \qquad m(e^{-\pi/\sqrt{n}}) = \sqrt{n}. \tag{15.6.14}$$

Thus, $1 - 2x_{1/n} = -(1 - 2x_n)$, and, from (15.6.11) and (15.6.14), we deduce that

$$6e^{-\pi/\sqrt{n}} \frac{m'}{m}(e^{-\pi/\sqrt{n}}) = P(e^{-2\pi/\sqrt{n}}) - nP(e^{-2\pi\sqrt{n}}) + 2n(1-2x_n)z_n^2. \tag{15.6.15}$$

The formula above is of central importance, because for twelve values of n, Ramanujan [239], [242, pp. 33–34] derived useful representations for $nP(q^{2n}) - P(q^2)$ in terms of theta functions.

We can eliminate $P(e^{-2\pi/\sqrt{n}})$ from (15.6.15) using the transformation formula for $P(q)$, which we now derive. Recall the transformation formula for Ramanujan's function $f(-q)$ given by [54, p. 43, Entry 27(iii)]

$$e^{-\alpha/12}\alpha^{1/4} f(-e^{-2\alpha}) = e^{-\beta/12}\beta^{1/4} f(-e^{-2\beta}), \tag{15.6.16}$$

where $\alpha\beta = \pi^2$, with α and β both positive. Taking the logarithm of both sides of (15.6.16), we find that

$$-\frac{\alpha}{12} + \frac{1}{4}\log\alpha + \sum_{k=1}^{\infty} \log(1 - e^{-2k\alpha}) = -\frac{\beta}{12} + \frac{1}{4}\log\beta + \sum_{k=1}^{\infty} \log(1 - e^{-2k\beta}). \tag{15.6.17}$$

Differentiating both sides of (15.6.17) with respect to α, we deduce that

$$-\frac{1}{12} + \frac{1}{4\alpha} + \sum_{k=1}^{\infty} \frac{2ke^{-2k\alpha}}{1 - e^{-2k\alpha}} = \frac{\beta}{12\alpha} - \frac{1}{4\alpha} - \sum_{k=1}^{\infty} \frac{(2k\beta/\alpha)e^{-2k\beta}}{1 - e^{-2k\beta}}. \tag{15.6.18}$$

Multiplying both sides of (15.6.18) by 12α and rearranging, we arrive at

$$6 - \alpha\left(1 - 24\sum_{k=1}^{\infty} \frac{ke^{-2k\alpha}}{1 - e^{-2k\alpha}}\right) = \beta\left(1 - 24\sum_{k=1}^{\infty} \frac{ke^{-2k\beta}}{1 - e^{-2k\beta}}\right). \tag{15.6.19}$$

Setting $\alpha = \pi/\sqrt{n}$, so that $\beta = \pi\sqrt{n}$, recalling the definition of $P(q)$, and rearranging slightly, we see that (15.6.19) takes the shape

$$\frac{6\sqrt{n}}{\pi} = P(e^{-2\pi/\sqrt{n}}) + nP(e^{-2\pi\sqrt{n}}). \tag{15.6.20}$$

Utilizing (15.6.20) in (15.6.15), we conclude that

$$6e^{-\pi/\sqrt{n}} \frac{m'}{m}(e^{-\pi/\sqrt{n}}) = \frac{6\sqrt{n}}{\pi} - 2nP(e^{-2\pi\sqrt{n}}) + 2n(1 - 2x_n)z_n^2. \tag{15.6.21}$$

Return to (15.6.5) and set $q = e^{-\pi\sqrt{n}}$. After (15.6.3), define

$$X_n = 4x_n(1 - x_n). \tag{15.6.22}$$

Thus, (15.6.5) takes the form

$$P(e^{-2\pi\sqrt{n}}) = \sum_{k=0}^{\infty}\{(1-2x_n)+3(1-2x_n)k\}A_k X_n^k. \tag{15.6.23}$$

Divide both sides of (15.6.21) by $2n$ and substitute $P(e^{-2\pi\sqrt{n}})$ from (15.6.23) into (15.6.21) to deduce that

$$\begin{aligned}\frac{3e^{-\pi/\sqrt{n}}}{n}\frac{m'}{m}(e^{-\pi/\sqrt{n}}) &= \frac{3}{\pi\sqrt{n}} - \sum_{k=0}^{\infty}\{(1-2x_n)+3(1-2x_n)k\}A_k X_n^k \\ &\quad + (1-2x_n)z_n^2 \\ &= \frac{3}{\pi\sqrt{n}} - 3\sum_{k=0}^{\infty}(1-2x_n)kA_k X_n^k,\end{aligned}$$

where we have used (15.6.2) with $q = e^{-\pi\sqrt{n}}$ and (15.6.22). Hence, we have derived the following general series representation for $1/\pi$.

Theorem 15.6.1. *Let $X_n = 4x_n(1-x_n)$ and recall that A_k is defined in* (15.6.3)*. Then for any positive integer n,*

$$\frac{1}{\pi\sqrt{n}} = \sum_{k=0}^{\infty}(1-2x_n)kA_k X_n^k + \frac{e^{-\pi/\sqrt{n}}}{n}\frac{m'}{m}(e^{-\pi/\sqrt{n}}). \tag{15.6.24}$$

Now, by the chain rule, (15.6.6), and (15.6.3),

$$q\frac{dm}{dq} = q\frac{dm}{dX}\frac{dX}{dx}\frac{dx}{dq} = \frac{dm}{dX}4(1-2x)x(1-x)z^2 = z^2X(1-2x)\frac{dm}{dX}. \tag{15.6.25}$$

Set $q = e^{-\pi/\sqrt{n}}$ in (15.6.25) and recall from (15.6.14) that $z_{1/n}^2 = nz_n^2$ and $m = \sqrt{n}$. Thus,

$$\begin{aligned}\frac{e^{-\pi/\sqrt{n}}}{2n}\frac{m'}{m}(e^{-\pi/\sqrt{n}}) &= \frac{z_n^2}{2\sqrt{n}}X_{1/n}(1-2x_{1/n})\frac{dm}{dX}\bigg|_{q=e^{-\pi/\sqrt{n}}} \\ &= \frac{1}{2\sqrt{n}}X_{1/n}(1-2x_{1/n})\frac{dm}{dX}\bigg|_{q=e^{-\pi/\sqrt{n}}}\sum_{k=0}^{\infty}A_k X_n^k,\end{aligned} \tag{15.6.26}$$

by (15.6.3). Substituting (15.6.26) into (15.6.24), we arrive at the following theorem.

Theorem 15.6.2. *Let $X = 4x(1-x)$ and $X_n = 4x_n(1-x_n)$. Let A_k be given by* (15.6.3)*. Then for any positive integer n,*

$$\frac{1}{\pi\sqrt{n}} = \sum_{k=0}^{\infty}(a_n + b_nk)A_k X_n^k, \tag{15.6.27}$$

where

$$a_n = \frac{1}{2\sqrt{n}} X_{1/n}(1-2x_{1/n}) \frac{dm}{dX}\bigg|_{q=e^{-\pi/\sqrt{n}}} \qquad \textit{and} \qquad b_n = 1-2x_n. \quad (15.6.28)$$

Our next task is to use Theorem 15.6.2 to establish three series for $1/\pi$ found on page 370 in Ramanujan's lost notebook. These three series are series (28)–(30) in Ramanujan's epic paper [239], [242, pp. 36–37]. Although these three series were established by the Borweins [86, Chapter 5], the proofs we provide here are different. In fact, our derivations are along the same lines as those of Ramanujan in [239], [242, p. 36]. In particular, we use representations for Ramanujan's function [239], [242, pp. 33–34]

$$f_n(q) := nP(q^{2n}) - P(q^2) \qquad (15.6.29)$$

for $n = 3, 7, 15$. (Ramanujan used the notation $f(n)$ instead of $f_n(q)$.) In [239], Ramanujan recorded representations for $f_n(q)$ for 12 values of n, but he gave no indication how these might be proved. These formulas are also recorded in Chapter 21 of Ramanujan's second notebook [243], and proofs may be found in [54]. The proofs given below can also be found in the paper [45] by N.D. Baruah and Berndt. In [239], Ramanujan stated 17 series for $1/\pi$, and Baruah and Berndt [45] established most of these series as well as many new series for $1/\pi$ using Ramanujan's ideas.

Entry 15.6.1 (p. 370). *Recall from* (15.6.3) *that*

$$A_k := \frac{(\frac{1}{2})_k^3}{k!^3}, \qquad k \geq 0.$$

Then

$$\frac{4}{\pi} = \sum_{k=0}^{\infty} (6k+1) A_k \frac{1}{4^k}. \qquad (15.6.30)$$

Proof. Let $n = 3$. Then, in the notation (15.6.29) [239], [242, p. 33], [54, p. 460, Entry 3(iii)],

$$f_3(q) = z(q)z(q^3)\left(1 + \sqrt{x(q)x(q^3)} + \sqrt{(1-x(q))(1-x(q^3))}\right). \qquad (15.6.31)$$

Set $q = e^{-\pi/\sqrt{3}}$ in (15.6.31) and use (15.6.14) to deduce that

$$f_3(e^{-\pi/\sqrt{3}}) = \sqrt{3}\left(1 + 2\sqrt{x_3(1-x_3)}\right) z_3^2. \qquad (15.6.32)$$

Recall the modular equation, due to A.M. Legendre and rediscovered by Ramanujan [243, Chapter 19, Entry 5(ii)], [54, p. 230],

$$\left\{x(q)x(q^3)\right\}^{1/4} + \left\{(1-x(q))(1-x(q^3))\right\}^{1/4} = 1. \qquad (15.6.33)$$

Setting $q = e^{-\pi/\sqrt{3}}$ in (15.6.33) and using (15.6.14), we find that

$$2\left\{x_3(1-x_3)\right\}^{1/4} = 1, \tag{15.6.34}$$

or, in the notation (15.6.22),

$$X_3 = 4x_3(1-x_3) = \frac{1}{4}. \tag{15.6.35}$$

Using (15.6.34) or (15.6.35) in (15.6.32), we deduce that

$$f_3(e^{-\pi/\sqrt{3}}) = \frac{3\sqrt{3}}{2} z_3^2. \tag{15.6.36}$$

In fact, x_3 is a singular modulus, and Ramanujan calculated this singular modulus in his notebooks [243], [57, p. 290]. Thus, from the aforementioned source or from (15.6.35),

$$1 - 2x_3 = \frac{\sqrt{3}}{2}. \tag{15.6.37}$$

Setting $n = 3$ in (15.6.15), recalling the definition (15.6.29), and employing (15.6.36) and (15.6.37), we deduce that

$$e^{-\pi/\sqrt{3}} \frac{m'}{m}(e^{-\pi/\sqrt{3}}) = \frac{1}{6}\left(-\frac{3\sqrt{3}}{2} + 6\frac{\sqrt{3}}{2}\right) z_3^2 = \frac{\sqrt{3}}{4} z_3^2. \tag{15.6.38}$$

Using (15.6.37), (15.6.38), (15.6.35), and (15.6.2) in (15.6.24) with $n = 3$, we find that

$$\frac{\sqrt{3}}{\pi} = \sum_{k=0}^{\infty}\left(\frac{3\sqrt{3}}{2}k + \frac{\sqrt{3}}{4}\right) A_k \frac{1}{4^k},$$

which is easily seen to be equivalent to (15.6.30). □

Entry 15.6.2 (p. 370). *If A_k, $k \geq 0$, is defined by* (15.6.3), *then*

$$\frac{16}{\pi} = \sum_{k=0}^{\infty} (42k+5) A_k \frac{1}{2^{6k}}. \tag{15.6.39}$$

Proof. We begin with a modular equation of degree 7,

$$\left\{x(q)x(q^7)\right\}^{1/8} + \left\{(1-x(q))(1-x(q^7))\right\}^{1/8} = 1, \tag{15.6.40}$$

due to C. Guetzlaff in 1834 but rediscovered by Ramanujan in Entry 19(i) of Chapter 19 of his second notebook [243], [54, p. 314]. Set $q = e^{-\pi/\sqrt{7}}$ in (15.6.40) and use (15.6.14) and (15.6.22) to deduce that

$$2\left\{x_7(1-x_7)\right\}^{1/8} = 1 \qquad \text{and} \qquad X_7 = \frac{1}{2^6}. \tag{15.6.41}$$

Ramanujan calculated the singular modulus x_7 in his first notebook [243], [57, p. 290], from which, or from (15.6.41), we easily can deduce that

$$1 - 2x_7 = \frac{3\sqrt{7}}{8}. \tag{15.6.42}$$

In the notation (15.6.29), from either [239], [242, p. 33] or [57, p. 468, Entry 5(iii)],

$$f_7(q) = 3z(q)z(q^7)\left(1 + \sqrt{x(q)x(q^7)} + \sqrt{(1-x(q))(1-x(q^7))}\right). \tag{15.6.43}$$

Putting $q = e^{-\pi/\sqrt{7}}$ in (15.6.43) and employing (15.6.14) and (15.6.41), we find that

$$f_7(e^{-\pi/\sqrt{7}}) = 3\sqrt{7}\left(1 + 2\sqrt{x_7(1-x_7)}\right)z_7^2 = 3\sqrt{7}\cdot\frac{9}{8}z_7^2. \tag{15.6.44}$$

Thus, by (15.6.15), (15.6.44), and (15.6.42),

$$6e^{-\pi/\sqrt{7}}\frac{m'}{m}(e^{-\pi/\sqrt{7}}) = -\frac{27\sqrt{7}}{8}z_7^2 + \frac{42\sqrt{7}}{8}z_7^2 = \frac{15\sqrt{7}}{8}z_7^2. \tag{15.6.45}$$

Then, using (15.6.24) with $n = 7$, and with the help of (15.6.42), (15.6.45), (15.6.41), and (15.6.2), we conclude that

$$\frac{3}{\sqrt{7}\pi} = \sum_{k=0}^{\infty}\left(\frac{9\sqrt{7}}{8}k + \frac{15\sqrt{7}}{14\cdot 8}\right)A_k\frac{1}{2^{6k}},$$

which is readily seen to be equivalent to (15.6.39). $\square$

Entry 15.6.3 (p. 370). *If A_k, $k \geq 0$, is defined by* (15.6.3), *then*

$$\frac{32}{\pi} = \sum_{k=0}^{\infty}\left((42\sqrt{5}+30)k + 5\sqrt{5} - 1\right)A_k\frac{1}{2^{6k}}\left(\frac{\sqrt{5}-1}{2}\right)^{8k}. \tag{15.6.46}$$

Proof. Recall that the Ramanujan–Weber class invariant G_n can be represented in terms of the singular moduli x_n by [57, p. 185]

$$G_n = \{4x_n(1-x_n)\}^{-1/24}. \tag{15.6.47}$$

Also recall that [57, p. 190]

$$G_{15} = 2^{1/4}\left(\frac{\sqrt{5}+1}{2}\right)^{1/3}. \tag{15.6.48}$$

From (15.6.47), (15.6.48), and (15.6.22), we can deduce that

$$X_{15} = \frac{1}{2^6}\left(\frac{\sqrt{5}-1}{2}\right)^8. \tag{15.6.49}$$

The singular modulus

$$x_{15} = \frac{1}{16}\left(\frac{\sqrt{5}-1}{2}\right)^4 (2-\sqrt{3})^2(4-\sqrt{15})$$

was calculated by Ramanujan and recorded in his first notebook [243], [57, p. 291], and so we can deduce that

$$1-2x_{15} = \frac{1}{32\sqrt{15}}(42\sqrt{5}+30). \tag{15.6.50}$$

(Of course, we can also deduce (15.6.50) from (15.6.49).) Next, from Entry 9(iii) in Chapter 21 of Ramanujan's second notebook [243], [54, p. 481], or from [239], [242, p. 34],

$$\begin{aligned} f_{15}(q) = z^2(q)z^2(q^{15})\Big(&\left\{1+\sqrt{x(q)x(q^{15})}+\sqrt{(1-x(q))(1-x(q^{15}))}\right\}^4 \\ &-1-\sqrt{x(q)x(q^{15})}-\sqrt{(1-x(q))(1-x(q^{15}))}\Big). \end{aligned} \tag{15.6.51}$$

Setting $q = e^{-\pi/\sqrt{15}}$ in (15.6.51) and using (15.6.14), (15.6.47), (15.6.48), and (15.6.49), we find that

$$\begin{aligned} f_{15}(e^{-\pi/\sqrt{15}}) &= \sqrt{15}\left(\left(1+2^{3/4}X_{15}^{1/8}\right)^4 - (1+\sqrt{X_{15}})\right) z_{15}^2 \\ &= \frac{3\sqrt{15}}{16}(11+9\sqrt{5})z_{15}^2. \end{aligned} \tag{15.6.52}$$

It follows from (15.6.15), (15.6.52), and (15.6.50) that

$$\begin{aligned} 6e^{-\pi/\sqrt{15}}\frac{m'}{m}(e^{-\pi/\sqrt{15}}) &= -\frac{3\sqrt{15}}{16}(11+9\sqrt{5})z_{15}^2 + \frac{30}{32\sqrt{15}}(42\sqrt{5}+30)z_{15}^2 \\ &= \left(\frac{75\sqrt{3}}{16} - \frac{3\sqrt{15}}{16}\right) z_{15}^2. \end{aligned} \tag{15.6.53}$$

Using (15.6.50), (15.6.53), (15.6.49), and (15.6.2) in (15.6.24) with $n=7$, we conclude that

$$\begin{aligned} \frac{3}{\sqrt{15}\pi} = \sum_{k=0}^{\infty}&\left(\frac{3}{32\sqrt{15}}(42\sqrt{5}+30)k + \frac{1}{30}\left(\frac{75\sqrt{3}}{16} - \frac{3\sqrt{15}}{16}\right)\right) \\ &\times A_k \frac{1}{2^{6k}}\left(\frac{\sqrt{5}-1}{2}\right)^{8k}. \end{aligned} \tag{15.6.54}$$

If we multiply both sides of (15.6.54) by $\frac{32}{3}\sqrt{15}$ and simplify, we obtain (15.6.46) to complete the proof. $\square$

The credit for returning to Ramanujan's ideas for deriving series for $1/\pi$ is due to Heng Huat Chan. The content of Section 15.6 is therefore almost entirely due to Chan, who sent detailed lecture notes to the second author of this book. These notes then inspired Baruah and Berndt to further elaborate the ideas of Ramanujan and Chan in a series of three papers [45], [46], [47]. In the first paper [45], they use Eisenstein series in the classical base to prove 13 of Ramanujan's original series, along with many other new series. In the second [46], they employ Eisenstein series in Ramanujan's cubic and quartic theories to prove five of Ramanujan's series and several new series for $1/\pi$. Lastly, in [47], they employ Ramanujan's ideas emphasizing Eisenstein series to derive many new series representations for $1/\pi^2$.

It does not appear to have been widely noticed that the first mathematician to have published a proof of a general formula for $1/\pi$ was S. Chowla [125], [124] [126, pp. 87–91, 116–119]. In particular, he established Entry 15.6.1. As indicated in the introduction, the Borwein brothers [86] first proved all 17 formulas for $1/\pi$ found in Ramanujan's paper [239], [242, pp. 23–39]. In a series of papers [87], [88], [89], [91], [92], they greatly extended Ramanujan's work, deriving a host of interesting formulas for $1/\pi$. The Chudnovskys [132]–[136] also amplified and explained Ramanujan's work. In [137], they also derived several hypergeometric-like series for π. The work of Berndt and Chan [64] gives a third approach for generalizing Ramanujan's series. General series formulas for $1/\pi$ have also been found by Berndt, Chan, and W.-C. Liaw [66], H.H. Chan, S.H. Chan, and Z.-G. Liu [103], H.H. Chan and Liaw [108], and H.H. Chan, Liaw, and V. Tan [109]. Further particular series for $1/\pi$ have been derived by H.H. Chan and H. Verrill [113], H.H. Chan and K.P. Loo [111], H.H. Chan and W. Zudilin [114], M.D. Rogers [249], and S. Cooper [141]. J. Guillera [163]–[168] has discovered some beautiful series for $1/\pi$ as well as for $1/\pi^2$. Further work has been accomplished by W. Zudilin [292]–[295], with the latter paper offering an interesting survey. Baruah, Berndt, and H.H. Chan [48] have written a survey paper delineating most of the research on series for $1/\pi$ since the publication of Ramanujan's paper [239], while also covering the use of such series by R.W. Gosper Jr. [160] and the Chudnovsky brothers [135] in calculating the digits of π.

16

Miscellaneous Results on Eisenstein Series

We collect in this chapter some miscellaneous results on Eisenstein series that do not fit into previous chapters.

16.1 A generalization of Eisenstein Series

On page 332 in [244], Ramanujan enigmatically records the following statement.

Entry 16.1.1 (p. 332).

$$``\frac{1^r}{e^{1^s x}-1}+\frac{2^r}{e^{2^s x}-1}+\frac{3^r}{e^{3^s x}-1}+\cdots, \tag{16.1.1}$$

where s is a positive integer and $r-s$ is any even integer."

(The statement (16.1.1) was poorly photocopied and is difficult to read.) If we set $q=e^{-x}$, we can write (16.1.1) in the equivalent form

$$\sum_{k=0}^{\infty}\frac{k^r q^{k^s}}{1-q^{k^s}}. \tag{16.1.2}$$

Thus, if $s=1$ and $r=2n-1$ is odd, (16.1.2) is a multiple of the classical Eisenstein series $E_{2n}(\tau)$, where $q=e^{2\pi i\tau}$ and $\operatorname{Im}\tau>0$. What does Ramanujan mean by (16.1.1)? We think that Ramanujan temporarily thought that a theory could be developed for these more general Eisenstein series that generalizes the classical theory. Because the series (16.1.2) do not live in either the elliptic or the modular world (except when $s=1$), such a theory indeed would be limited. We have carefully examined Ramanujan's theory of Eisenstein series as he developed it in [240], [242, pp. 136–162] to discern whether it can be generalized. (See also [59, Chapter 4], where details are more completely given.) If so, we would need to make the following definitions.

Define, for each nonnegative integer r and positive integer s,

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$$S_{r,s} := -\frac{B_{r+1}}{2(r+1)} + \sum_{k=1}^{\infty} \frac{k^r q^{k^s}}{1-q^{k^s}}. \tag{16.1.3}$$

When $s = 1$, $S_{r,1} = S_r$, in Ramanujan's notation [240, equation (9)], [59, equation (4.1.4)]. Also define for each nonnegative integer r and positive integer s,

$$\Psi_{r,s}(q) := \sum_{n=1}^{\infty} \frac{n^r q^{n^s}}{(1-q^{n^s})^2}. \tag{16.1.4}$$

In Ramanujan's notation [240, equations (23), (24)], [59, equations (4.1.3), (4.2.7)] $\Psi_{r,1}(q) = \Phi_{1,r}(q)$. Unfortunately, although the proofs of the recurrence relations involving S_r and $\Psi_{r,s}$ in [240] do not depend on the theory of either elliptic or modular functions, we have not been able to see how they can be generalized to the functions (16.1.3) and (16.1.4).

16.2 Representations of Eisenstein Series in Terms of Elliptic Function Parameters

The following six formulas appear in Ramanujan's notebooks, but in Ramanujan's individual notation, instead of the classical notation used here in the lost notebook. In particular, in [243], Ramanujan sets $z := 2K/\pi$, where K is the complete elliptic integral of the first kind, $\alpha = k^2$, where k, $0 < k < 1$, is the modulus, and $1 - \alpha = k'$, where $k' = \sqrt{1-k^2}$ is the complementary modulus.

Entry 16.2.1 (p. 367). *In the notation above,*

$$\begin{aligned}
Q(q) &= \left(\frac{2K}{\pi}\right)^4 \left(1 + 14k^2 + k^4\right), \\
Q(q^4) &= \left(\frac{K}{\pi}\right)^4 \left(1 + 14k'^2 + k'^4\right), \\
Q(q^2) &= \left(\frac{2K}{\pi}\right)^4 \left(1 - (kk')^2\right), \\
R(q) &= \left(\frac{2K}{\pi}\right)^6 (1+k^2)(1 - 34k^2 + k^4), \\
R(q^4) &= \left(\frac{K}{\pi}\right)^6 (1+k'^2)\left(1 - 34k'^2 + k'^4\right), \\
R(q^2) &= \left(\frac{2K}{\pi}\right)^6 (k'^2 - k^2)\left(1 + \tfrac{1}{2}(kk')^2\right).
\end{aligned}$$

These six identities are, respectively, Entry 13(iii), Entry 13(v), Entry 13(i), Entry 13(iv), Entry 13(vi), and Entry 13(ii), in Chapter 17 of Ramanujan's second notebook [243]. See [54, pp. 126–128] for their statements in Ramanujan's notation and for proofs.

16.3 Values of Certain Eisenstein Series

On page 334, Ramanujan defines a certain function of s involving Eisenstein series, for which he calculates values at integral arguments of multiples of 4. Ramanujan's motivation for these calculations is unclear. For increasing values of s, the calculations become increasingly laborious. Lacking Ramanujan's patience and arithmetic skills, we turned to *Mathematica*.

Entry 16.3.1 (p. 334). *Define the function* S_s *by*

$$\sum_{k=1}^{\infty} \frac{k^{1-s}}{e^{2k\pi}-1} = S_s - \frac{1}{2}\zeta(s-1) + \frac{3}{4\pi}\zeta(s) + \frac{\pi}{12}\zeta(s-2), \tag{16.3.1}$$

where ζ *denotes the Riemann zeta function. Then, if* n *is a positive integer,*

$$S_{4n} = 0, \quad S_0 = \frac{1}{4\pi}, \quad S_4 = -\frac{\pi^3}{360}, \quad S_8 = 0, \quad S_{12} = -\frac{\pi^{11}}{232186500},$$
$$S_{16} = \frac{\pi^{15}}{21470872500}, \quad S_{20} = \frac{191\pi^{19}}{398240480137500},$$
$$S_{24} = \frac{907\pi^{23}}{184177171143590625}.$$

Proof. First, let $s = -4n$. Since $\zeta(-2n) = 0$ for each positive integer n [271, p. 19], Ramanujan's claim that $S_{-4n} = 0$ reduces to

$$\sum_{k=1}^{\infty} \frac{k^{4n+1}}{e^{2\pi k}-1} = -\frac{\zeta(-4n-1)}{2} = \frac{B_{4n+2}}{8n+4}, \tag{16.3.2}$$

where B_n, $n \geq 0$, denotes the nth Bernoulli number, and where we have used the well-known formula [271, p. 19]

$$\zeta(1-2n) = -\frac{B_{2n}}{2n}.$$

The identity (16.3.2) has been proved many times in the literature, with the proof of J.W.L. Glaisher [155] in 1889 being the oldest proof known to us. The identity can also be found in Ramanujan's second notebook [243, Chapter 12, Corollary (iv)], [53, p. 262]. For references to several other proofs given in the literature, see [53, pp. 261–262].

When $s = 0$, Ramanujan's claim takes the form

$$\sum_{k=1}^{\infty} \frac{k}{e^{2\pi k}-1} = \frac{1}{24} - \frac{1}{8\pi}, \tag{16.3.3}$$

where we have used the facts, $\zeta(-2) = 0$, $\zeta(-1) = -\frac{1}{2}B_2 = -\frac{1}{12}$, and $\zeta(0) = -\frac{1}{2}$ [271, p. 19]. Ramanujan posed (16.3.3) as a problem in the *Journal of the Indian Mathematical Society* [238], [242, p. 326], [75, p. 240]. The identity

(16.3.3) also appears as an example in Section 8 of Chapter 14 in his second notebook [243], [53, p. 256]. To the best of our knowledge, the first appearance of (16.3.3) in the literature is in a paper by O. Schlömilch [252] in 1877. References to several other proofs can be found in [53, p. 256].

Putting $s = 4n$ in (16.3.1), we find that

$$\sum_{k=1}^{\infty} \frac{k^{1-4n}}{e^{2k\pi}-1} = S_{4n} - \frac{1}{2}\zeta(4n-1) + \frac{3}{4\pi}\zeta(4n) + \frac{\pi}{12}\zeta(4n-2). \tag{16.3.4}$$

For the remaining six values, namely, $n = 1, 2, 3, 4, 5, 6$ in (16.3.4), we employ the special case $\alpha = \beta = \pi$ of a famous identity for $\zeta(2n+1)$, $n \geq 1$, found in Ramanujan's second notebook [243], [53, pp. 275–276, Entry 21(i)]. Replacing n by $2n+1$ in the aforementioned identity, we deduce that, for $n \geq 1$,

$$\frac{1}{2}\zeta(4n-1) + \sum_{k=1}^{\infty} \frac{k^{1-4n}}{e^{2\pi k}-1} = -2^{4n-3}\pi^{4n-1} \sum_{k=0}^{2n} (-1)^k \frac{B_{2k}}{(2k)!} \frac{B_{4n-2k}}{(4n-2k)!}. \tag{16.3.5}$$

This particular case of Ramanujan's famous identity is due to M. Lerch in 1901 [202]. Using Euler's famous formula

$$\zeta(2n) = \frac{(2\pi)^{2n} B_{2n}}{2(2n)!}, \qquad n \geq 1,$$

comparing (16.3.4) with (16.3.5), and dividing both sides of each identity by π^{4n-1}, we find that

$$\frac{S_{4n}}{\pi^{4n-1}} = 3\frac{2^{4n-3}B_{4n}}{(4n)!} - \frac{2^{4n-5}B_{4n-2}}{3(4n-2k)!} - 2^{4n-3}\sum_{k=0}^{2n}(-1)^k \frac{B_{2k}}{(2k)!}\frac{B_{4n-2k}}{(4n-2k)!}, \tag{16.3.6}$$

where n is any positive integer. Using *Mathematica*, we calculated the right-hand side of (16.3.6) for $n = 1, 2, 3, 4, 5, 6$ and found that Ramanujan's claims in Entry 16.3.1 are correct for each of these six values of n. □

The identities (16.3.2), (16.3.3), and (16.3.5) (and its aforementioned generalization) also appear in an incomplete handwritten manuscript published with Ramanujan's lost notebook [244, pp. 318–321] and examined by Berndt [58]. This manuscript will also be examined in [33].

16.4 Some Elementary Identities

Entry 16.4.1 (p. 367). *If s is a positive integer and B_n denotes the nth Bernoulli number, then*

$$(2^s-1)\frac{B_s}{2s} + \sum_{k=1}^{\infty} \frac{k^{s-1}q^k}{1+(-q)^k}$$

$$= 2^s \left\{ \frac{B_s}{2s} - \sum_{k=1}^{\infty} \frac{k^{s-1}q^{4k}}{1-q^{4k}} \right\} - \left\{ \frac{B_s}{2s} - \sum_{k=1}^{\infty} \frac{k^{s-1}q^{k}}{1-q^{k}} \right\} \tag{16.4.1}$$

and

$$(2^s-1)\frac{B_s}{2s} + \sum_{k=1}^{\infty} \frac{k^{s-1}q^k}{1+(-q)^k} = 2^s \left\{ \frac{B_s}{2s} + \sum_{k=1}^{\infty} \frac{k^{s-1}q^{4k}}{1-q^{4k}} \right\}$$
$$+ \left\{ \frac{B_s}{2s} + \sum_{k=1}^{\infty} \frac{k^{s-1}q^{k}}{1-q^{k}} \right\} - 2\left\{ \frac{B_s}{2s} + \sum_{k=1}^{\infty} \frac{k^{s-1}q^{2k}}{1-q^{2k}} \right\}. \tag{16.4.2}$$

Proof. Canceling the expressions involving Bernoulli numbers, we find that (16.4.1) is equivalent to the identity

$$\sum_{k=1}^{\infty} \frac{(2k)^{s-1}q^{2k}}{1+q^{2k}} + \sum_{k=0}^{\infty} \frac{(2k+1)^{s-1}q^{2k+1}}{1-q^{2k+1}}$$
$$= -2^s \sum_{k=1}^{\infty} \frac{k^{s-1}q^{4k}}{1-q^{4k}} + \sum_{k=1}^{\infty} \frac{k^{s-1}q^{k}}{1-q^{k}}$$
$$= -2^{s-1} \sum_{k=1}^{\infty} \frac{k^{s-1}q^{2k}}{1-q^{2k}} + 2^{s-1} \sum_{k=1}^{\infty} \frac{k^{s-1}q^{2k}}{1+q^{2k}} + \sum_{k=1}^{\infty} \frac{k^{s-1}q^{k}}{1-q^{k}}. \tag{16.4.3}$$

Cancel the first sum on the far left-hand side with the second sum on the far right-hand side above. If the even- and odd-indexed terms in the last sum on the far right-hand side of (16.4.3) are separated, we see that the proposed identity (16.4.3) is trivial.

To prove (16.4.2), we first cancel all expressions involving Bernoulli numbers. We then easily find that (16.4.2) is equivalent to the identity

$$\sum_{k=0}^{\infty} \frac{(2k+1)^{s-1}q^{2k+1}}{1+q^{2k+1}} + 2^{s-1} \sum_{k=1}^{\infty} \frac{k^{s-1}q^{2k}}{1-q^{2k}}$$
$$= 2^{s-1} \sum_{k=1}^{\infty} \frac{k^{s-1}q^{2k}}{1-q^{2k}} - 2^{s-1} \sum_{k=1}^{\infty} \frac{k^{s-1}q^{2k}}{1+q^{2k}}$$
$$+ \sum_{k=1}^{\infty} \frac{k^{s-1}q^{k}}{1-q^{k}} - 2\sum_{k=1}^{\infty} \frac{k^{s-1}q^{2k}}{1-q^{2k}}. \tag{16.4.4}$$

The second sum on the left-hand side cancels with the first sum on the right-hand side in (16.4.4). Moving the second sum on the right-hand side to the left-hand side, we find that (16.4.4) is equivalent to the identity

$$\sum_{k=1}^{\infty} \frac{k^{s-1}q^{k}}{1+q^{k}} = \sum_{k=1}^{\infty} \frac{k^{s-1}q^{k}}{1-q^{k}} - 2\sum_{k=1}^{\infty} \frac{k^{s-1}q^{2k}}{1-q^{2k}}. \tag{16.4.5}$$

Using the elementary identity

$$\frac{x}{1+x} = \frac{x}{1-x} - \frac{2x^2}{1-x^2}$$

above, we see that (16.4.5) is trivial, and so the proof of (16.4.2) is also complete. $\square$

Location Guide

For each page of Ramanujan's lost notebook on which we have discussed or proved entries in this book, we provide below a list of those entries. If (n) appears after an entry, the entry has n parts.

Provenance

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G.E. Andrews, [6]
G.E. Andrews, [7]
G.E. Andrews, [9]
B.C. Berndt and A.J. Yee, [79]
Padmavathamma, [225]

Chapter 2

None

Chapter 3

None

Chapter 4

A.V. Sills, [261]

Chapter 5

G.E. Andrews, [22]
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Chapter 6

R.P. Agarwal, [4]
G.E. Andrews, [21]
G.E. Andrews, [28]

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Chapter 7

G.E. Andrews, [20]
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G.E. Andrews, J. Jiménez-Urroz, and K. Ono, [36]

Chapter 8

S.H. Son, [266]
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Chapter 9

B.C. Berndt, H.H. Chan, S.-Y. Kang, and L.-C. Zhang, [65]
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Chapter 10

B.C. Berndt, H.H. Chan, and A. Zaharescu, [68]
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Chapter 11

B.C. Berndt and P.R. Bialek, [61]
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Chapter 12

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Chapter 13

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S. Raghavan and S.S. Rangachari, [233]

Chapter 14

B.C. Berndt and A.J. Yee, [77]

Chapter 15

N.D. Baruah and B.C. Berndt, [45]
B.C. Berndt and H.H. Chan, [64]

Chapter 16

None

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